SUBJECT

DATE

2.1

(a.)

$$\Psi(x,t) = \psi(x) e^{-iE/\hbar\, t} = \psi(x) e^{-i(E_0+i\Gamma)t/\hbar}, \text{ where } E = E_0 + i\Gamma$$

where $E_0, \Gamma \in \mathbb{R}$

$$\Psi(x,t) = \psi(x) e^{\Gamma t/\hbar} e^{-iE_0/\hbar\, t}$$

$$\Psi^*(x,t) = \psi(x) e^{\Gamma t/\hbar} e^{+iE_0/\hbar\, t}$$

$$1 = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\, dx = \int_{-\infty}^{\infty} \left[\psi(x) e^{\Gamma t/\hbar} \cancel{e^{+iE_0/\hbar\, t}} \cdot \psi(x) e^{\Gamma t/\hbar} \cancel{e^{-iE_0/\hbar\, t}}\right] dx$$

$$1 = e^{2\Gamma/\hbar\, t} \int_{-\infty}^{\infty} \psi^2(x)\, dx$$

If this eqn. is to hold for all time, $t[-\infty, \infty]$, the time-dependent term must be constant. That must mean $2\Gamma/\hbar = 0$, which means $\Gamma = 0$ ($2/\hbar$ is constant). If $\Gamma = 0$, then $E = E_0 + i\Gamma$ is $E = E_0$ ~~and~~, with $E_0 \in \mathbb{R}$. Therefore, $E \in \mathbb{R} \rightarrow E$ is ~~real~~ must be real.

QED

(b.) Taking $E, V \in \mathbb{R}$: if ψ satisfies the time-independent SE,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

then so does its complex conjugate:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + V\psi^* = E\psi^*$$

Next, if ψ_1 and ψ_2 are both solns to the time-independent SE, then any linear combination of ψ_1 and ψ_2 are also solns: $\psi_3 \equiv c_1\psi_1 + c_2\psi_2$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} + V\psi_3 = \frac{-\hbar^2}{2m}\left(c_1\frac{d^2\psi_1}{dx^2} + c_2\frac{d^2\psi_2}{dx^2}\right) + V(c_1\psi_1 + c_2\psi_2)$$

$$= c_1\underbrace{\left(\frac{-\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right)}_{E\psi_1} + c_2\underbrace{\left(\frac{-\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right)}_{E\psi_2}$$

$$= c_1 E\psi_1 + c_2 E\psi_2 = E\underbrace{(c_1\psi_1 + c_2\psi_2)}_{=\psi_3}$$

$$= E\psi_3$$

Thus, ~~for~~ any soln, ψ, of the time-independent SE implies ① ψ^* is also a soln ~~of the~~ and ② any linear combination of solns is also a soln. As such, the real linear combinations $\psi + \psi^*$ and $i(\psi - \psi^*)$ satisfy the time-independent SE, as long as ψ is a soln and $V, E \in \mathbb{R}$. Therefore, ~~any~~ ψ can always be constructed from two real solns (noting that if ψ is already real, the 2nd soln will be 0).

(c.) If $V(-x) = V(x)$ and $\psi(x)$ is a soln to the time-independent SE:

$x \rightarrow -x$ (Δ of variables)

$$\frac{d^2}{d(-x)^2} = \frac{d^2}{dx^2} \text{ (even function), so } \frac{d^2\psi(-x)}{d(-x)^2} = \frac{d^2\psi(-x)}{dx^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + \underbrace{V(-x)}_{V(x)}\psi(-x) = E\psi(-x)$$

But this is just the time-independent SE for $-x$ over $x[-\infty, \infty]$, which we have already established as a valid soln [i.e. $\psi(x)$ is a soln for $x[-\infty,\infty]$ so $\psi(-x)$ for $-x(-\infty, \infty)$].

$\therefore$ $\psi_+(x) \equiv \psi(x) + \psi(-x)$ (which is even)

and $\psi_-(x) = \psi(x) - \psi(-x)$ (which is odd)

both satisfy the time-independent SE.

Finally, we can write $\psi(x) = \frac{1}{2}\left(\psi_+(x) + \psi_-(x)\right)$, thus showing any soln, $\psi(x)$, of the time-independent SE can be expressed as a linear combo of even and odd solns.

QED.

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2.4 $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$

$$\langle x\rangle = \int_0^L x\frac{2}{L}\sin^2\left(\frac{n\pi}{L}x\right)dx \qquad z' = \frac{n\pi}{L}x \quad dz' = \frac{n\pi}{L}dx$$

$$\langle x\rangle = \frac{2}{L}\int_0^{n\pi}\left(\frac{L}{n\pi}\right)^2 z'\sin^2(z')dz'$$

$$= \frac{2L}{n^2\pi^2}\int_0^{n\pi} z'\sin^2 z'\,dz' = \frac{2L}{n^2\pi^2}\int_0^{n\pi}\frac{z'}{2}(1-\cos 2z')dz'$$

$$= \frac{\cancel{2}L}{\cancel{2}n^2\pi^2}\left[\frac{z^2}{2}\Big|_0^{n\pi} - \int_0^{n\pi} z'\cos 2z'\,dz'\right] \qquad \begin{matrix} z' & \oplus & \cos 2z' \\ 1 & \ominus & \frac{1}{2}\sin 2z' \\ & & -\frac{1}{4}\cos 2z' \end{matrix}$$

$$= \frac{L}{n^2\pi^2}\left[\frac{n^2\pi^2}{2} - \frac{z}{2}\sin 2z\Big|_0^{n\pi} - \frac{1}{4}\cos 2z\Big|_0^{n\pi}\right]$$

$$= \frac{L}{2} - 0 + 0 - \cancel{\frac{1}{4}(1-1)}$$

$$\boxed{\langle x\rangle = L/2}$$

$$\langle x^2\rangle = \frac{2}{L}\int_0^L x^2\sin^2\left(\frac{n\pi}{L}x\right)dx \qquad z' = \frac{n\pi}{L}x \quad dz' = \frac{n\pi}{L}dx$$

$$= \frac{2}{L}\int_0^{n\pi}\left(\frac{L}{n\pi}\right)^3 z'^2\sin^2(z')dz' = \frac{2L^2}{n^3\pi^3}\int_0^{n\pi}\frac{z'^2}{2}(1-\cos 2z')dz'$$

$$\left.\begin{matrix} z'^2 & \oplus & \cos 2z' \\ 2z' & \ominus & \frac{1}{2}\sin 2z' \\ 2 & \oplus & -\frac{1}{4}\cos 2z' \\ 0 & & -\frac{1}{8}\sin 2z' \end{matrix}\right\} \quad \frac{z'^2}{2}\sin 2z' + \frac{z'}{2}\cos 2z' - \frac{1}{4}\sin 2z'$$

$$\langle x^2\rangle = \frac{L^2}{n^3\pi^3}\left[\frac{z^3}{3} - \frac{z^2}{2}\sin 2z - \frac{z}{2}\cos 2z + \frac{1}{4}\sin 2z\right]_0^{n\pi}$$

$$= \frac{L^2}{n^3\pi^3}\left(\frac{n^3\pi^3}{3} - \cancel{\frac{n^2\pi^2}{2}\sin 2n\pi}^{\,0} - \frac{n\pi}{2}\cancel{\cos 2n\pi}^{\,+1} + \cancel{\frac{1}{4}\sin 2n\pi}^{\,0} - 0 + 0 + 0 - 0\right)$$

$$\langle x^2\rangle = L^2\left[\frac{1}{3}-\frac{1}{2}\frac{1}{n^2\pi^2}\right]$$

$$\sigma_x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2} = \left(L^2\left[\frac{1}{3}-\frac{1}{2(n\pi)^2}\right]-\frac{L^2}{4}\right)^{1/2}$$

$$= L\sqrt{\frac{1}{3}-\frac{1}{2(n\pi)^2}-\frac{1}{4}} = L\sqrt{\frac{1}{12}-\frac{1}{2(n\pi)^2}} = \boxed{\frac{L}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}}$$

$$\boxed{\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0 \quad \text{since } \langle x\rangle \text{ is a constant}}$$

$$\langle p^2\rangle = \int_0^L \psi_n^*\left(\frac{\hbar}{i}\right)^2\frac{d^2\psi_n}{dx^2}\,dx = -\hbar^2\int_0^L \psi_n^*\frac{d^2\psi_n}{dx^2}\,dx$$

Recall one of our starting relations (i.e., the time-independent SE):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} = E_n\psi_n \longrightarrow -\frac{d^2\psi_n}{dx^2} = \frac{2m}{\hbar^2}E_n\psi_n$$

$$= -\hbar^2\int_0^L \psi_n^*\boxed{\frac{d^2\psi_n}{dx^2}}\,dx \Rightarrow \frac{2m\cancel{\hbar^2}}{\cancel{\hbar^2}}E_n\underbrace{\int_0^L \psi_n^*\psi_n\,dx}_{\delta_{mn}=1 \text{ for } n=m} = 2mE_n$$

$$\langle p^2\rangle = \cancel{2m}\left(\frac{n^2\pi^2\hbar^2}{\cancel{2m}L^2}\right) = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left(\frac{n\pi\hbar}{L}\right)^2} = \frac{n\pi\hbar}{L}$$

$$\sigma_x\sigma_p = \left(\frac{L}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}\right)\left(\frac{n\pi\hbar}{L}\right) = \frac{n\pi\hbar}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}$$

$$\boxed{\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3}-2}}$$

$n=1$ limit: $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} > \frac{\hbar}{2}$

$(1.14)\hbar/2$

$n\to$ large limit: $\frac{\hbar}{2\sqrt{3}}n\pi \sim \sigma_x\sigma_p$

Smallest value of $\sigma_x\sigma_p$ achieve for $n=1$

SUBJECT

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2.5 $\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$

(a.)

$$|\Psi|^2 = \Psi^*\Psi = A^2\left[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2\right]$$

$$1 = \int_{-\infty}^{\infty}|\Psi|^2 dx = A^2\int_{-\infty}^{\infty}\left[\psi_1^*\psi_1 + \cancel{\psi_1^*\psi_2} + \cancel{\psi_2^*\psi_1} + \psi_2^*\psi_2\right]dx$$

1   0 b/c of orthogonality   1

$$1 = 2A^2 \longrightarrow \boxed{A = \frac{1}{\sqrt{2}}}$$

(b.)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{\frac{-iE_1}{\hbar}t} + \psi_2(x)e^{\frac{-iE_2}{\hbar}t}\right]$$

where $\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$, $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)$, and
$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2 = \hbar\omega_0 n^2$, where $\omega_0 = \frac{\pi^2\hbar}{2mL^2}$

$$\boxed{\Psi(x,t) = \sqrt{\frac{1}{L}}\left[\underbrace{\sin\left(\frac{\pi}{L}x\right)}_{\alpha} + \underbrace{\sin\left(\frac{2\pi}{L}x\right)}_{\beta}e^{-3i\omega_0 t}\right]e^{-i\omega_0 t}}$$

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) = \frac{1}{L}\left[\left(\alpha + \beta e^{+3i\omega_0 t}\right)\cancel{e^{+i\omega_0 t}}\right]\left[\left(\alpha + \beta e^{-3i\omega_0 t}\right)\cancel{e^{-i\omega_0 t}}\right]$$

$$= \frac{1}{L}\left(\alpha^2 + \alpha\beta e^{-3i\omega_0 t} + \alpha\beta e^{+3i\omega_0 t} + \beta^2\right)$$

$$= \frac{1}{L}\left(\alpha^2 + \beta^2 + \alpha\beta\left[e^{3i\omega_0 t} + e^{-3i\omega_0 t}\right]\right)$$

$$\cos\theta = \frac{1}{2}\left[e^{+i\theta} + e^{-i\theta}\right]$$

$$\boxed{|\Psi(x,t)|^2 = \frac{1}{L}\left[\sin^2\left(\frac{\pi}{L}x\right) + 2\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)\cos(3\omega_0 t) + \sin^2\left(\frac{2\pi}{L}x\right)\right]}$$

(c.) $\langle x\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat{x}\,\Psi(x,t)dx = \int_0^L \Psi^*(x,t)\,x\,\Psi(x,t)dx$

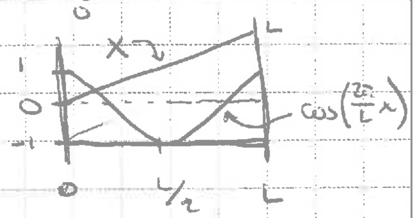
$$\langle x\rangle = \frac{1}{L}\int_0^L\left[x\sin^2\left(\frac{\pi}{L}x\right) + 2x\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)\cos(3\omega_0 t) + x\sin^2\left(\frac{2\pi}{L}x\right)\right]dx$$

$$= \frac{1}{L}\left\{\underbrace{\int_0^L\left(\frac{x - x\cos\left(\frac{2\pi}{L}x\right)}{2}\right)dx}_{1} + \underbrace{\int_0^L 2x\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)\cos(3\omega_0 t)dx}_{2} + \underbrace{\int_0^L\left(\frac{x - x\cos\left(\frac{4\pi}{L}x\right)}{2}\right)dx}_{3}\right\}$$

$\langle\cancel{x}\rangle =$

1. $$\int_0^L x\sin^2\left(\frac{\pi}{L}x\right)dx = \int_0^L\left(\frac{x - x\cos\left(\frac{2\pi}{L}x\right)}{2}\right)dx = \frac{x^2}{4} - \int_0^L \frac{x\cos\left(\frac{2\pi}{L}x\right)}{2}dx$$

x		$\cos\left(\frac{2\pi}{L}x\right)$
1	$+$	$\frac{L}{2\pi}\sin\left(\frac{2\pi}{L}x\right)$
0	$-$	$\frac{-L^2}{4\pi^2}\cos\left(\frac{2\pi}{L}x\right)$



$$\frac{x^2}{4} - \frac{1}{2}\left[\frac{L}{2\pi}\sin\left(\frac{2\pi}{L}x\right) + \frac{L^2}{4\pi^2}\cos\left(\frac{2\pi}{L}x\right)\right]_0^L = \frac{x^2}{4} - \frac{1}{2}\left(0 - 0 + \frac{L^2}{4\pi^2}(1-1)\right)$$

$$= \frac{x^2}{4}$$

3. $$\int_0^L x\sin^2\left(\frac{2\pi}{L}x\right)dx$$ $\frac{2\pi}{L}x \Rightarrow$ very similar to $\frac{\pi}{L}x$ in the above integral

$$= \frac{x^2}{4} - \int_0^L \frac{x\cos\left(\frac{4\pi}{L}x\right)}{2}dx = \frac{x^2}{4} - \frac{1}{2}\left[\frac{L}{4\pi}\sin\left(\frac{4\pi}{L}x\right) + \frac{L^2}{16\pi^2}\cos\left(\frac{4\pi}{L}x\right)\right]_0^L$$

$$= \frac{x^2}{4} - \frac{1}{2}\left(0 - 0 + \frac{L^2}{16\pi^2}(1-1)\right) = \frac{x^2}{4}$$

2. $$\int_0^L x\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)\cos 3\omega_0 t\, dx = \cos 3\omega_0 t\underbrace{\int_0^L x\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)dx}$$

we only need to worry about this integral

$$\int_0^L x\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)dx = \frac{1}{2}\int_0^L x\left[\cos\left(\frac{\pi}{L}x\right) - \cos\left(\frac{3\pi}{L}x\right)\right]dx$$

$$\frac{1}{2}\left[\cos\left(\frac{2\pi-\pi}{L}x\right) - \cos\left(\frac{2\pi+\pi}{L}\right)\right]$$

$$= \frac{1}{2}\left[\frac{L^2}{\pi^2}\cos\left(\frac{\pi}{L}x\right) + \frac{Lx}{\pi}\sin\left(\frac{\pi}{L}x\right) - \frac{L^2}{9\pi^2}\cos\left(\frac{3\pi}{L}x\right) - \frac{Lx}{3\pi}\sin\left(\frac{3\pi}{L}x\right)\right]$$

$$= \frac{1}{2}\left[\frac{L^2}{\pi^2}(\underbrace{\cancel{-1}-1}_{-2}) - \frac{L^2}{9\pi^2}(\underbrace{\cancel{-1}-1}_{-2})\right] = \frac{-L^2}{\pi^2}\left(1 - \frac{1}{9}\right) = \frac{-8L^2}{9\pi^2}$$

Putting everything together:

$$\langle x\rangle = \frac{1}{\cancel{L}}\left[\underbrace{\frac{L^2}{4}}_{1} - \underbrace{2\cos(3\omega_0 t)\frac{8L^2}{9\pi^2}}_{2} + \underbrace{\frac{L^2}{4}}_{3}\right] = \boxed{\frac{L}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega_0 t)\right]}$$

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(d.) $\langle p \rangle = m\dfrac{d\langle x\rangle}{dt}$

$$\langle p\rangle = \frac{md}{dt}\left(\frac{L}{2}\left(1-\frac{32}{9\pi^2}\cos 3\omega_0 t\right)\right) = \frac{\cancel{32}^{16} m}{{}_3\cancel{9}\pi^2}\cancel{3}\omega_0 \sin(3\omega_0 t)\frac{L}{\cancel{2}}$$

$$= \frac{16 m\omega_0 L}{3\pi^2}\sin(3\omega_0 t) = \frac{\cancel{16}^{8}\cancel{m}\cancel{L}}{3\cancel{\pi^2}}\frac{\cancel{\pi^2}\hbar}{\cancel{2}\cancel{m}\cancel{L^2}_{L}}\sin(3\omega_0 t) = \boxed{\frac{8\hbar}{3L}\sin(3\omega_0 t)}$$

(e.) If you measured the energy of this particle, you would either obtain $E_1\left(=\frac{\pi^2\hbar^2}{2mL^2}\right)$ or $E_2\left(=\frac{2\pi^2\hbar^2}{2mL^2}\right)$

with equal probability, $1/2$. Here, I'm using $|A|^2$ which is really: $|c_1|^2 = |c_2|^2 = |A|^2$ in this problem or

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x) = A(\psi_1(x)+\psi_2(x))$$

where we found A to be $\frac{1}{\sqrt{2}}$.

$$\langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2 E_n = \frac{1}{2}(E_1+E_2) = \frac{1}{2}\left(\frac{\pi^2\hbar^2+2\pi^2\hbar^2}{2mL^2}\right)$$

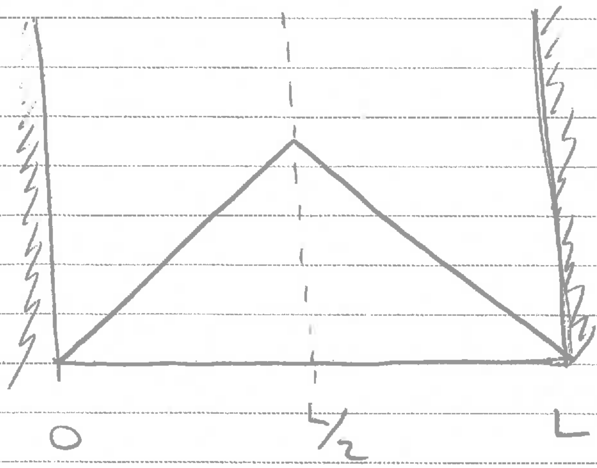
$\langle H\rangle = \dfrac{5\pi^2\hbar^2}{4mL^2}$ (i.e., the expectation value of the energy is the average of E_1 and E_2)

SUBJECT

DATE (8)

2.7 $\Psi(x,0) = \begin{cases} Ax & x[0, L/2] \\ A(L-x) & x[L/2, L] \end{cases}$

(a.)



$$1 = A^2 \underbrace{\int_0^{L/2} x^2 dx}_{} + A^2 \underbrace{\int_{L/2}^{L} (L-x)^2 dx}_{}$$

symmetric about $L/2$

$$1 = 2A^2 \int_0^{L/2} x^2 dx = 2A^2 \frac{x^3}{3}\Big|_0^{L/2}$$

$$1 = 2A^2\left(\frac{L^3}{24}\right) \rightarrow \boxed{A = \frac{2\sqrt{3}}{L\sqrt{L}} \text{ or } \frac{2}{L}\sqrt{\frac{3}{L}}}$$

(b.) $\Psi_n(x,t) = \underbrace{A\Psi_n(x)\varphi_n(t)}_{} = \frac{2}{L}\sqrt{\frac{3}{L}} \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-iE_n}{\hbar}t}$

where $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$

known

unknown

$$c_n = \int \Psi_n^*(x)\Psi(x,0)\,dx$$

$$= \frac{2}{L^2}\sqrt{6}\left[\int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx + \int_{L/2}^{L} (L-x)\sin\left(\frac{n\pi}{L}\right)dx\right]$$

Two cases:

Case 1, n is even

When n is even, the Ψ_n's about $L/2$ are odd.

Since $\Psi(x,0)$ about $L/2$ ~~is~~ even, $\int_0^{L} \psi_n^*(x)\Psi(x,0)\,dx = 0$ for all values of n that are even.

Case 2; n is odd

Like we saw in class for $\psi_1(x)$, if the integrand [i.e., $\psi_n^*(x)\Psi(x,0)$] is even about $L/2$, we can eliminate the $L/2$ to L integral and multiple the 0 to $L/2$ integral by 2.

$$c_n = \frac{4\sqrt{6}}{L^2}\int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\begin{array}{ccc} x & & \sin\left(\frac{n\pi}{L}x\right) \\ & \oplus & \\ 1 & & -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right) \\ & \ominus & \\ 0 & & -\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{L}x\right) \end{array}$$

$$= \frac{4\sqrt{6}}{L^2}\left[-\frac{L}{n\pi}x\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\right]_0^{L/2}$$

$$= \frac{4\sqrt{6}}{L^2}\left[\underbrace{-\frac{L}{n\pi}\left(\frac{L}{2}\cos\left(\frac{n\pi}{2}\right) - 0\right)}_{0 \text{ for all } n} + \left(\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - 0\right)\right]$$

$$= \frac{4\sqrt{6}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \quad \text{for } n \text{ odd} \Rightarrow (-1)^{\frac{n-1}{2}}\frac{4\sqrt{6}}{(n\pi)^2}$$

$$\therefore\; c_n = \begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^{\frac{n-1}{2}}\dfrac{4\sqrt{6}}{(n\pi)^2} & \text{for } n \text{ odd} \end{cases}$$

Putting this altogether:

$$\boxed{\Psi(x,t) = \frac{4\sqrt{6}}{(\pi)^2}\sqrt{\frac{2}{L}}\sum_{n=1,3,5,7,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\frac{\sin\left(\frac{n\pi}{L}x\right)}{L^2}e^{-\frac{iE_n}{\hbar}t}}, \text{ where } E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

(c.) $P_1 = |c_1|^2 = \boxed{\left[\frac{4\sqrt{6}}{\pi^2}\right]^2 = 0.986}$

(d.) $$\langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n = \sum_{n=1,3,5,\ldots}^{\infty}\frac{\cancel{16}^{8}\cdot 6}{\cancel{n^4}\,\cancel{\pi^4}}\,\frac{\cancel{n^2}\cancel{\pi^2}\hbar^2}{\cancel{2}mL^2}$$

$$= \frac{48\cancel{\pi^2}\hbar^2}{\cancel{\pi^4}mL^2}\underbrace{\sum_{n=1,3,5,\ldots}^{\infty}(1/n^2)}_{\pi^2/8} = \frac{\cancel{48}^{6}\hbar^2}{\cancel{\pi^2}mL^2}\frac{\cancel{\pi^2}}{\cancel{8}} = \boxed{\frac{6\hbar^2}{mL^2}}$$

SUBJECT

DATE (10)

$Lx - x^2$

$L - 2x$

2.9

$$\Psi(x,0) = Ax(L-x) \qquad x[0,L]$$

$$\langle H \rangle = \int \Psi(x,0)^* \hat{H} \Psi(x,0)\, dx$$

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} \quad \text{for } V(x)=0 \text{ over } x[0,L]$$

$$\langle H \rangle = \int_0^L A^2 x(L-x)\left(-\frac{\hbar^2}{2m}\right)(-2)\,dx$$

$$\uparrow \frac{d^2\psi}{dx^2}$$

$$= \frac{\hbar^2}{m}\int_0^L A^2 x(L-x)\,dx$$

$$= \frac{\hbar^2}{m}A^2\left[\frac{Lx^2}{2}\Big|_0^L - \frac{x^3}{3}\Big|_0^L\right] = \frac{\hbar^2}{m}A^2\left[\frac{L^3}{2} - \frac{L^3}{3}\right]$$

$$\langle H \rangle = \frac{\hbar^2}{m}\frac{L^3}{6}A^2$$

$$1 = A^2\int_0^L x^2(L-x)^2\,dx = A^2\int_0^L (x^2L^2 - 2x^3L + x^4)\,dx$$

$$= A^2\left[\frac{L^2x^3}{3} - \frac{Lx^4}{2} + \frac{x^5}{5}\right]_0^L = A^2\left(L^5\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]\right)$$

$$\frac{10-15+6}{30} = \frac{1}{30}$$

$$\frac{30}{L^5} = A^2$$

$$\frac{1}{L^2}\sqrt{\frac{30}{L}} = A$$

$$\therefore \langle H \rangle = \frac{\hbar^2}{m}\frac{\cancel{L^3}}{\cancel{6}}\frac{\cancel{30}^{\,5}}{L^5} = \boxed{\frac{5\hbar^2}{mL^2}}$$

SUBJECT

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6.

$10\ cm^{-1} = 1000\ \mu m \longleftrightarrow \frac{1240\ nm\text{-}eV}{10^6\ nm} = 1.24\ meV$

$250\ cm^{-1} = 40\ \mu m \longleftrightarrow \frac{1240\ nm\text{-}eV}{4\times10^4\ nm} = 31\ meV$

$1500\ cm^{-1} = 6.67\ \mu m \longleftrightarrow \frac{1240\ nm\text{-}eV}{6.67\ \mu m} = 186\ meV$

7. $P_n(1/a) = \frac{2}{L}\int_0^{L/a} \sin^2\left(\frac{n\pi}{L}x\right)dx$

$= \frac{2}{L}\int_0^{L/a} \frac{1}{2}\left(1-\cos\left(\frac{2n\pi}{L}x\right)\right)dx$

$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{\cancel{L}}\left[\frac{\cancel{L}}{a} - \sin\left(\frac{2n\pi}{L}x\right)\frac{\cancel{L}}{2n\pi}\Big|_0^{L/a}\right]$

$\boxed{P_n(1/a) = \frac{1}{a} - \frac{1}{2n\pi}\sin\left(\frac{2n\pi}{a}\right)}$

As $n \to \infty$, $P_n(1/a) \to 1/a$ (classical result)