Tips, Hints, and Tricks for Problem 4.6.

Identify two cases: the first for $l > l'$ and the second $l = l'$. For both, integrate by parts l times. In the $l > l'$ case, the remaining integral can be solved if you carefully consider differentiating the term $\left(x^2-1\right)^{l'}$ $l - l'$ times. The boundary terms in both cases can be easily evaluated by showing that you will always have a $\left(x^2-1\right)$ term left over after $l - n$ differentiations, where n is the n^{th} boundary term, with n ranging from 1 to l.

Regarding the $l = l'$ case, after dealing with the boundary terms after l IBPs (exactly the same as the $l > l'$ case), you are left with $\frac{(-1)^l}{(2^l l!)^2}\int_{-1}^{+1}\left(x^2-1\right)^l\left(\frac{d}{dx}\right)^{2l}\left(x^2-1\right)^l dx$. Doing that $2l$ differentiation of $\left(x^2-1\right)^l$ is straightforward, as is moving in the $(-1)^l$ term inside the integral to give $\left(1-x^2\right)^l$. Next, make a substitution: $x = 2u - 1$. We want to recast the integral into the form of the beta function, $B(r,s)$:

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)}, \tag{1}$$

where $\Gamma(x)$ is the gamma function.