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①

2.20

(a.)

$$f(x) = \sum_{n=0}^{\infty}\left[a_n \sin\left(\frac{n\pi}{L}x\right) + b_n \cos\left(\frac{n\pi}{L}x\right)\right]$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$f(x) = b_0 + \sum_{n=1}^{\infty} \frac{a_n}{2i}\left(e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x}\right) + \sum_{n=1}^{\infty}\frac{b_n}{2}\left(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}\right)$$

$$= b_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{i\frac{n\pi}{L}x} + \sum_{n=1}^{\infty}\left(-\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{-i\frac{n\pi}{L}x}$$

$$c_0 = b_0 \qquad c_n = \tfrac{1}{2}(-ia_n + b_n) \quad \text{for } n = 1, 2, 3, \ldots$$

$$c_n = \tfrac{1}{2}(ia_n + b_n) \quad \text{for } n = -1, -2, -3, \ldots$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$ QED

(b.)

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

$$\int_{-L}^{L} f(x) e^{-i\frac{m\pi}{L}x}\,dx = \int_{-L}^{L} e^{-i\frac{m\pi}{L}x}\left(\sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}\right)dx$$

$$\int_{-L}^{L} f(x) e^{-i\frac{m\pi}{L}x}\,dx = \sum_{-\infty}^{\infty} c_n \underbrace{\int_{-L}^{L} e^{-i\frac{\pi}{L}(m-n)x}\,dx}_{①}$$

For $m \neq n$; ① is equal to 0:

$$\int_{-L}^{L} e^{-i\frac{\pi}{L}(m-n)x}\,dx = i\frac{L}{\pi}\frac{1}{m-n}\left[e^{-i\frac{\pi}{L}(m-n)x}\right]_{-L}^{L}$$

$m-n = p$, where $p$ is an integer, so: $\frac{iL}{\pi}\frac{1}{m-n}\left[e^{-i\pi p} - e^{i\pi p}\right]$

$$= \frac{iL}{\pi}\frac{1}{p}\left[\cos(-\pi p) - \cos(\pi p)\right] = 0$$

For $m \neq n$, $c_n = 0$.

For $m = n$:

$$\int_{-L}^{L} f(x) e^{-i\frac{m\pi}{L}x}\, dx = c_n \boxed{\int_{-L}^{L} dx} = 2L$$

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi}{L}x}\, dx = c_n}$$ QED

(c.) $k = \frac{n\pi}{L}$, $F(k) = \sqrt{\frac{2}{\pi}} L c_n$, $\Delta k = \frac{\pi}{L}\Delta n = \frac{\pi}{L}$ for $\Delta n = 1$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi/L\, x}$$ [from part (a)]

$$= \sum_{-\infty}^{\infty} c_n e^{ikx} \frac{L}{\pi}\Delta k$$ (substituting $\frac{kL}{\pi}$ for $n$ and $\frac{L\Delta k}{\pi}$ for $\Delta n$).

$$= \sum_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{L} F(k) e^{ikx} \frac{L}{\pi}\Delta k$$ (substituting in $\sqrt{\frac{\pi}{2}}\frac{1}{L}F(k)$ for $c_n$)

$$\boxed{f(x) = \sqrt{\frac{1}{2\pi}} \sum_{-\infty}^{\infty} F(k) e^{ikx} \Delta k}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi}{L}x}\, dx$$ [from part (b.)]

$$\sqrt{\frac{\pi}{2}}\frac{1}{\cancel{L}} F(k) = \frac{1}{2\cancel{L}}\int_{-L}^{L} f(x) e^{-ikx}\, dx$$ (substituting $\sqrt{\frac{\pi}{2}}\frac{1}{L}F(k)$ for $c_n$ and $k$ for $\frac{n\pi}{L}$)

$$\boxed{F(k) = \sqrt{\frac{1}{2\pi}}\int_{-L}^{L} f(x) e^{-ikx}\, dx}$$ QED

(d.) $L \to \infty$, $k = \frac{n\pi}{L}$ becomes a continuous variable

$\Delta k = \frac{\pi}{L}\Delta n \to dk$

$$f(x) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx}\, dk$$

$$F(k) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

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2.22

(a.) $\Psi(x,0) = Ae^{-ax^2}$

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = A^2 \sqrt{\frac{\pi}{2a}}$$

$$\boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}}$$

(b.)
$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \int_{-\infty}^{\infty} e^{-y^2 + \frac{b^2}{4a}} \frac{dy}{\sqrt{a}} = \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy = \boxed{\sqrt{\frac{\pi}{a}}\, e^{b^2/4a}}$$

$y \equiv \sqrt{a}\left[x + \frac{b}{2a}\right]$, $ax^2 + bx = y^2 - \frac{b^2}{4a}$ ; $dy = \sqrt{a}\,dx$

Let's first find $\phi(k)$

$$\phi(k) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx = A\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx$$

$a = a$ and $b = ik$, so;

$$\phi(k) = A\sqrt{\frac{1}{2\pi}} \sqrt{\frac{\pi}{a}}\, e^{\frac{(ik)^2}{4a}} = \left(\frac{2a}{\pi}\right)^{1/4} \left(\frac{1}{2a}\right)^{1/2} e^{-\frac{k^2}{4a}}$$

$$\boxed{\phi(k) = \left(\frac{1}{2\pi a}\right)^{1/4} e^{-\frac{k^2}{4a}}}$$

Next, we substitute in our explicit form of $\phi(k)$ to find $\Psi(x,t)$

$$\Psi(x,t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$$

$$= \sqrt{\frac{1}{2\pi}} \left(\frac{1}{2\pi a}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + ikx - i\frac{\hbar k^2}{2m} t} dk = \sqrt{\frac{1}{2\pi}} \left(\frac{1}{2\pi a}\right)^{1/4} \int_{-\infty}^{\infty} e^{-k^2\left(\frac{1}{4a} + \frac{i\hbar}{2m} t\right) + ikx} dk$$

Here $a = \frac{1}{4a} + \frac{i\hbar}{2m} t$ $\quad b = -ix$

$$\int_{-\infty}^{\infty} e^{-\left(k^2\left(\frac{1}{4a} + \frac{i\hbar}{2m} t\right) - ikx\right)} dk = \left(\frac{\pi}{\frac{1}{4a} + \frac{i\hbar}{2m} t}\right)^{1/2} e^{\frac{-x^2}{\frac{1}{a} + \frac{2i\hbar}{m} t}} = \left(\frac{4a\pi}{1 + 2\frac{ia\hbar}{m} t}\right)^{1/2} e^{\frac{-ax^2}{1 + 2\frac{ia\hbar}{m} t}}$$

$$\Psi(x,t) = \sqrt{\frac{1}{2\pi}}\left(\frac{1}{2\pi a}\right)^{1/4}\sqrt{\frac{4a\pi}{1+\frac{2ia\hbar}{m}t}}\ \exp\left(\frac{-ax^2}{1+\frac{2ia\hbar}{m}t}\right)$$

$$\boxed{\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\left(\frac{1}{1+\frac{2ia\hbar}{m}t}\right)^{1/2}\exp\left(\frac{-ax^2}{1+\frac{2ia\hbar}{m}t}\right)}$$

(c.) $|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$

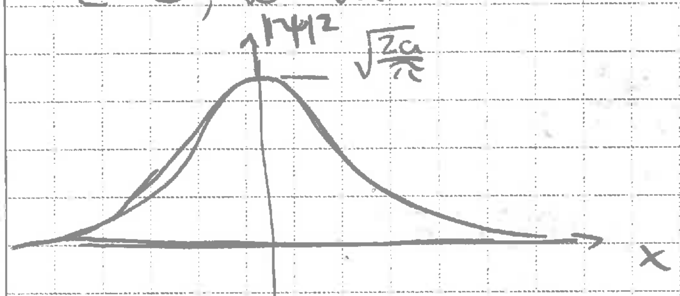
$$= \left(\frac{2a}{\pi}\right)^{1/2}\left(\frac{1}{1-\frac{2ia\hbar}{m}t}\right)^{1/2}\exp\left(\frac{-ax^2}{1-\frac{2ia\hbar}{m}t}\right)\left(\frac{1}{1+\frac{2ia\hbar}{m}t}\right)^{1/2}\exp\left(\frac{-ax^2}{1+\frac{2ia\hbar}{m}t}\right)$$

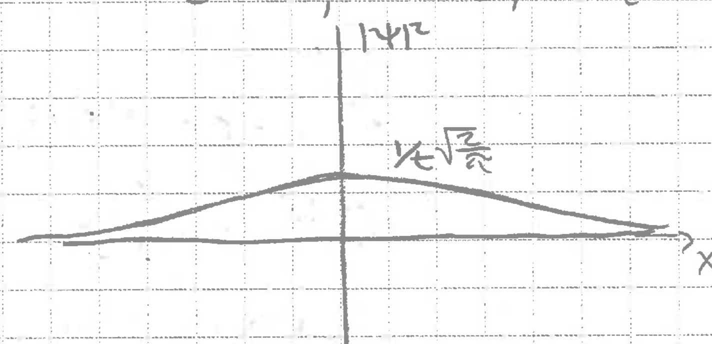
Let's define $w$ as: $w \equiv \sqrt{\frac{a}{1+\left(\frac{2\hbar a}{m}t\right)^2}}$

$$= \left(\frac{2a}{\pi}\right)^{1/2}\left(\frac{1}{1+\frac{4a^2\hbar^2}{m^2}t^2}\right)^{1/2}\exp\left(-ax^2\left[\frac{1+\cancel{\frac{2ia\hbar}{m}t}+1-\cancel{\frac{2ia\hbar}{m}t}}{1+\frac{4a^2\hbar^2}{m^2}t^2}\right]\right)$$

$$\boxed{|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\,w\exp(-2w^2x^2)}$$

$t=0$, $w=\sqrt{a}$ $\qquad$ $t\to\infty$, $w\to\frac{1}{t}$, $|\Psi|^2 \sim \frac{1}{t}e^{-\frac{2x^2}{t^2}}$




As $t$ becomes large, $|\Psi|^2$ broadens and flattens as a function of $x$.

(d.) $\langle x\rangle = \int_{-\infty}^{\infty}\Psi^* x\Psi\,dx = 0$ b/c integral is odd about 0.

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0$$

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$$\langle x^2\rangle = \int_{-\infty}^{\infty}\psi^* x^2\psi\, dx = w\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx$$

$$\frac{d}{d\lambda}\underbrace{\left[\int_{-\infty}^{\infty} e^{-\lambda x^2} dx\right]}_{\sqrt{\frac{\pi}{\lambda}}} \Rightarrow \frac{d}{d\lambda}\left(\sqrt{\frac{\pi}{\lambda}}\right) = \frac{d}{d\lambda}\left[\int_{-\infty}^{\infty} e^{-\lambda x^2} dx\right]$$

$$-\frac{1}{2}\sqrt{\pi}\,\lambda^{-3/2} = -\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx$$

$$\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} = \int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx$$

$\lambda \equiv 2w^2$ in our case, so:

$$\langle x^2\rangle = \cancel{w}\sqrt{\frac{2}{\pi}}\,\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}} = \boxed{\frac{1}{4w^2}}$$

$$\langle p^2\rangle = -\hbar^2\int\psi^*\frac{d^2}{dx^2}\psi\, dx$$

$$\psi = \left(\frac{2a}{\pi}\right)^{1/4}\left(\frac{1}{1+\frac{2ia\hbar}{m}t}\right)^{1/2}\exp\left(\frac{-ax^2}{1+\frac{2ia\hbar}{m}t}\right)$$

$$\beta = \frac{2a\hbar}{m}t$$

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

$$\psi = A\left(\frac{1}{1+i\beta}\right)^{1/2}\exp\left(\frac{-ax^2}{1+i\beta}\right)$$

$$\frac{d^2}{dx^2}\psi = \frac{d}{dx}\left(A\left(\frac{1}{1+i\beta}\right)^{1/2}\left[-2x\frac{a}{1+i\beta}\right]\exp\left(-\frac{ax^2}{1+i\beta}\right)\right)$$

$$= A\left(\frac{1}{1+i\beta}\right)^{1/2}\left[\frac{-2a}{1+i\beta} + \frac{4a^2x^2}{(1+i\beta)^2}\right]\exp\left(\frac{-ax^2}{1+i\beta}\right)$$

$$\langle p^2\rangle = -\hbar^2 A^2\underbrace{\left(\frac{1}{1+\beta^2}\right)^{1/2}}_{\frac{w}{\sqrt{a}}}\int_{-\infty}^{\infty}\left(\frac{-2a}{1+i\beta}+\frac{4a^2x^2}{(1+i\beta)^2}\right)\exp(-2w^2x^2)\,dx$$

$$= 2\hbar^2A^2\frac{wa}{1+i\beta}\int_{-\infty}^{\infty}\left(1-\frac{2ax^2}{1+i\beta}\right)e^{-2w^2x^2}dx$$

$$= 2\hbar^2A^2\frac{\cancel{w}\sqrt{a}}{1+i\beta}\left(\sqrt{\frac{\pi}{2w^2}} - \frac{2a}{1+i\beta}\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right) = 2\hbar^2\sqrt{\frac{2a}{\cancel{\pi}}}\frac{\sqrt{a}}{1+i\beta}\left(1-\frac{a}{1+i\beta}\frac{1}{2w^2}\right)\sqrt{\frac{\cancel{\pi}}{2}}$$

$\langle p^2\rangle = 2\hbar^2 \frac{a}{1+i\beta}\left(1 - \frac{a}{1+i\beta}\frac{1}{2w^2}\right)$ (6)

$$w = \sqrt{\frac{a}{(1+i\beta)(1-i\beta)}} \Rightarrow 1 - \frac{a}{1+i\beta}\frac{1}{2w^2} = 1 - \frac{\cancel{a}}{\cancel{1+i\beta}}\frac{\cancel{(1+i\beta)}(1-i\beta)}{\cancel{a}}\frac{1}{2}$$

$$= 1 - \frac{1-i\beta}{2} = \frac{1}{2}(1+i\beta)$$

$$\langle p^2\rangle = 2\hbar^2 \frac{a}{\cancel{1+i\beta}}\frac{1}{\cancel{2}}\cancel{(1+i\beta)} = \boxed{a\hbar^2}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{4w^2}} = \boxed{\frac{1}{2w}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{a\hbar^2} = \boxed{\hbar\sqrt{a}}$$

(e.) $\sigma_x\sigma_p = \frac{\hbar}{2w}\sqrt{a} = \frac{\hbar}{2}\frac{\cancel{\sqrt{a}}}{\sqrt{\frac{\cancel{a}}{1+\frac{4\hbar^2a^2}{m^2}t^2}}} = \boxed{\frac{\hbar}{2}\sqrt{1+\frac{4\hbar^2a^2}{m^2}t^2}}$

At $t=0$, $\sigma_x\sigma_p = \frac{\hbar}{2}$ (uncertainty threshold). As $t>0$, $\sigma_x\sigma_p$ increases, growing as $t$ for large $t$. Minimum uncertainty is achieved @ $t=0$.

2.25 $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$

$\langle x\rangle = 0$ (odd integral); $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0$

$$\langle x^2\rangle = \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty} x^2 e^{-\frac{2m\alpha|x|}{\hbar^2}}\,dx \rightarrow \frac{2m\alpha}{\hbar^2}\int_0^\infty x^2 e^{-\frac{2m\alpha x}{\hbar^2}}\,dx$$

$\beta \equiv \frac{2m\alpha}{\hbar^2}$

$$\langle x^2\rangle = \frac{2m\alpha}{\hbar^2}\int_0^\infty x^2 e^{-\beta x}\,dx$$

| | | |
|---|---|---|
| $x^2$ | $\oplus$ | $e^{-\beta x}$ |
| $2x$ | $\ominus$ | $\frac{-1}{\beta}e^{-\beta x}$ |
| $2$ | $\oplus$ | $\frac{1}{\beta^2}e^{-\beta x}$ |
| $0$ | | $\frac{-1}{\beta^3}e^{-\beta x}$ |

$$= \frac{2m\alpha}{\hbar^2}\left[\frac{-x^2}{\beta} - \frac{2x}{\beta^2} - \frac{2}{\beta^3}\right]e^{-\beta x}\Big|_0^\infty$$

$$= \underbrace{\frac{2m\alpha}{\hbar^2}}_{\beta}\frac{2}{\beta^3} = \frac{2}{\beta^2} = \boxed{\frac{\hbar^4}{2m^2\alpha^2}}$$

$$\sigma_x = \sqrt{\langle x^2\rangle} = \boxed{\frac{\hbar^2}{m\alpha\sqrt{2}}}$$

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$$\langle p^2\rangle = -\hbar^2 \int \psi^* \frac{d^2\psi}{dx^2}\, dx$$

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar} \begin{cases} -\frac{m\alpha}{\hbar^2} e^{-\frac{m\alpha}{\hbar^2}x} & x \geq 0 \\ \frac{m\alpha}{\hbar^2} e^{\frac{m\alpha}{\hbar^2}x} & x \leq 0 \end{cases} \to \left(\frac{\sqrt{m\alpha}}{\hbar^2}\right)^3 \left[ -\Theta(x) e^{-\frac{m\alpha}{\hbar^2}x} + \Theta(-x) e^{\frac{m\alpha}{\hbar^2}x} \right]$$

step function

$$\frac{d^2\psi}{dx^2} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[ -\delta(x) e^{-\frac{m\alpha}{\hbar^2}x} + \frac{m\alpha}{\hbar^2}\Theta(x) e^{-\frac{m\alpha}{\hbar^2}x} \right.$$

$$\left. -\delta(-x) e^{\frac{m\alpha}{\hbar^2}x} + \frac{m\alpha}{\hbar^2}\Theta(-x) e^{\frac{m\alpha}{\hbar^2}x} \right]$$

$$= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[ -2\delta(x) + \frac{m\alpha}{\hbar^2} e^{-\frac{m\alpha}{\hbar^2}|x|} \right]$$

Here $\delta(x) = \delta(-x)$ and $\delta(x) e^{-\frac{m\alpha}{\hbar^2}x} = \cancel{e^0}^{\,1}\,\delta(x)$.
Also, $\Theta(-x) + \Theta(x) = 1$

$$\langle p^2\rangle = -\hbar^2 \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^4 \left[ \int_{-\infty}^{\infty} -2\delta(x) e^{-\frac{m\alpha|x|}{\hbar^2}}\, dx + \frac{m\alpha}{\hbar^2} \int_{-\infty}^{\infty} e^{-\frac{2m\alpha}{\hbar^2}|x|}\, dx \right]$$

$$= \frac{m^2\alpha^2}{\hbar^2} \left[ 2 - \frac{2m\alpha}{\hbar^2} \int_0^{\infty} e^{-\frac{2m\alpha}{\hbar^2}x}\, dx \right]$$

$$= \frac{m^2\alpha^2}{\hbar^2} \left[ 2 - \underbrace{\frac{2m\alpha}{\hbar^2} \left(-\frac{\hbar^2}{2m\alpha}\right)(0-1)}_{1} \right] = \boxed{\left(\frac{m\alpha}{\hbar}\right)^2}$$

$$\sigma_p = \sqrt{\langle p^2\rangle} = \boxed{\frac{m\alpha}{\hbar}}$$

$$\sigma_x \sigma_p = \frac{\hbar^2}{\cancel{m\alpha}\sqrt{2}} \frac{\cancel{m\alpha}}{\cancel{\hbar}} = \frac{\hbar}{\sqrt{2}} = \frac{\sqrt{2}\,\hbar}{2} > \frac{\hbar}{2}$$ ✓

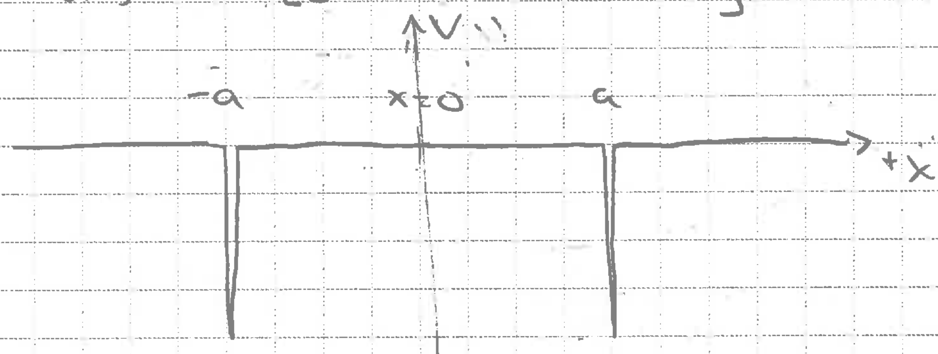
2.26
$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$F(k) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = \boxed{\sqrt{\frac{1}{2\pi}}}$$

$$f(x) = \delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{1}{2\pi}}}_{F(k)} e^{ikx}\, dk = \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk \quad \text{QED}}$$

2.27 $\quad V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$

(a.)




(b.) Sols are both even and odd

Even solns:
$$\psi(x) = \begin{cases} Ae^{-kx} & x > a \\ B(e^{kx} + e^{-kx}) & x(-a,a) \\ Ae^{kx} & x < -a \end{cases}$$

At a, we have wavefunction continuity:
$$Ae^{-ka} = B(e^{ka} + e^{-ka}) \rightarrow A = B(e^{2ka} + 1)$$

Derivative @ a is discontinuous:
$$\Delta \frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2} \psi(a)$$

$$-kAe^{-ka} - B(ke^{ka} - ke^{-ka}) = -\frac{2m\alpha}{\hbar^2} Ae^{-ka}$$

$$A + B(e^{2ka} - 1) = \frac{2m\alpha}{\hbar^2 k} A \rightarrow B(e^{2ka} - 1) = A\left(\frac{2m\alpha}{\hbar^2 k} - 1\right)$$

For the odd solns:

$$\psi(x) = \begin{cases} Ae^{-kx} & x>a \\ B(e^{kx}-e^{-kx}) & x\in(-a,a) \\ -Ae^{kx} & x<-a \end{cases}$$

Continuity @ a:

$$Ae^{-ka} = B(e^{ka}-e^{-ka}) \rightarrow A = B(e^{2ka}-1)$$

$$\Delta\left(\frac{d\psi}{dx}\right)\Big|_a = -\frac{2m\alpha}{\hbar^2}\psi(a)$$

$$-kAe^{-ka} - kB(e^{ka}+e^{-ka}) = -\frac{2m\alpha}{\hbar^2}Ae^{-ka}$$

$$B(e^{ka}+e^{-ka}) = \left(\frac{2m\alpha}{\hbar^2 k}-1\right)Ae^{-ka}$$

$$B(e^{2ka}+1) = A\left(\frac{2m\alpha}{\hbar^2 k}-1\right)$$

substituting for this

$$(e^{2ka}+1) = (e^{2ka}-1)\left(\frac{2m\alpha}{\hbar^2 k}-1\right)$$

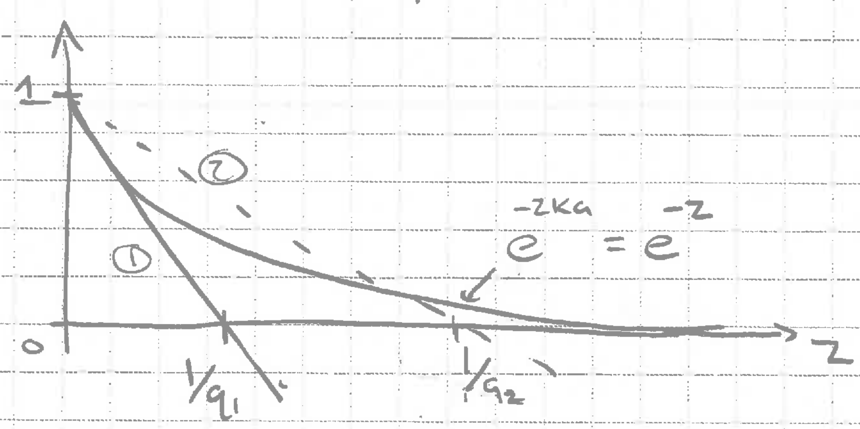
$$e^{2ka} = e^{2ka}\left(\frac{2m\alpha}{\hbar^2 k}-1\right) - \frac{2m\alpha}{\hbar^2 k}$$

$$\frac{2m\alpha}{\hbar^2 k} - 2 = \frac{2m\alpha}{\hbar^2 k}e^{-2ka}$$

$$\boxed{1 - \frac{\hbar^2 k}{m\alpha} = e^{-2ka}}$$

$z \equiv 2ka \qquad q = \frac{\hbar^2}{2am\alpha}$

$$1 - qz = e^{-z}$$




We will have zero odd bound states if $1/q$ is small (i.e., if $\alpha$ is small, since $1/q$ is proportional to $\alpha$). If $1/q$ is large,

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Substituting $B(e^{2Ka}+1)$ for A:

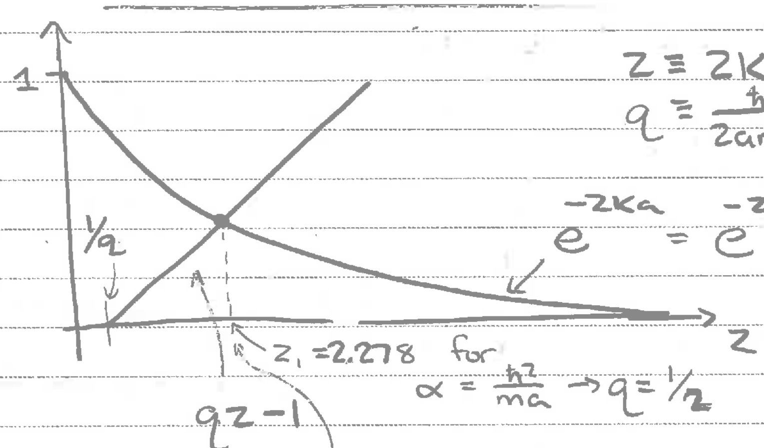
$$\cancel{B}(e^{2Ka}-1) = \cancel{B}(e^{2Ka}+1)\left(\frac{2m\alpha}{\hbar^2 K}-1\right)$$

$$e^{2Ka}\cancel{-1} = \frac{2m\alpha}{\hbar^2 K}\cancel{-1} + e^{2Ka}\left(\frac{2m\alpha}{\hbar^2 K}-1\right)$$

$$e^{2Ka} = \frac{m\alpha}{\hbar^2 K}(e^{2Ka}+1)$$

$$e^{2Ka}\left(1-\frac{m\alpha}{\hbar^2 K}\right) = \frac{m\alpha}{\hbar^2 K}$$

$$\boxed{\frac{\hbar^2 K}{m\alpha}-1 = e^{-2Ka}}$$




$$\left.\begin{array}{l} z \equiv 2Ka \\ q \equiv \frac{\hbar^2}{2am\alpha} \end{array}\right\} \begin{array}{l} e^{-2Ka} \Rightarrow e^{-z} \\ \frac{\hbar^2 K}{m\alpha}-1 \Rightarrow qz-1 \end{array}$$

$z_2 = 0.739$ for $\alpha = \frac{\hbar^2}{4ma} \to q = 2$.

In both cases, there is only one bnd state.

$$K^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \to E = -\frac{\hbar^2}{2m}\frac{z^2}{4a^2} = \frac{-\hbar^2}{8ma^2}z^2$$

For $\alpha = \frac{\hbar^2}{ma}$, $z_1 = 2.278$.

$$E = -\frac{\hbar^2}{8ma^2}(2.278)^2 = \boxed{-0.615\frac{\hbar^2}{ma^2}}$$

For $\alpha = \frac{\hbar^2}{4ma}$, $z_2 = 0.739$

$$\boxed{E = -0.0682\frac{\hbar^2}{ma^2}}$$

We always have one even bnd state.

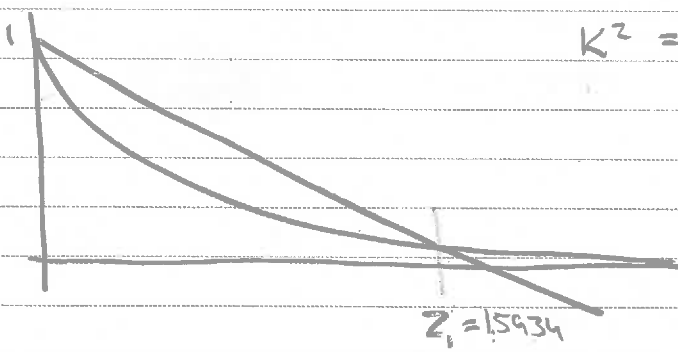
then we will have one odd bnd state, $z=0$ is not a soln, since it implies $K=0$. which makes $\psi$ non-normalizable,

At $z=0$, $e^{-z}$ has a slope of $-1$ $\left(\frac{de^{-z}}{dz}\Big|_{z=0} = -1\right)$, while $1-qz$ has a slope of $-q$. If $-q>-1$, then one soln's; if $-q \leq -1$, then no soln.

$$\therefore \quad q<1 \rightarrow \frac{\hbar^2}{2am} < \alpha \quad \text{one odd bnd states}$$

$$q \geq 1 \rightarrow \frac{\hbar^2}{2am} \geq \alpha \quad \text{no odd bnd state}$$

$\alpha = \frac{\hbar^2}{ma} \longrightarrow$ one odd bound state soln

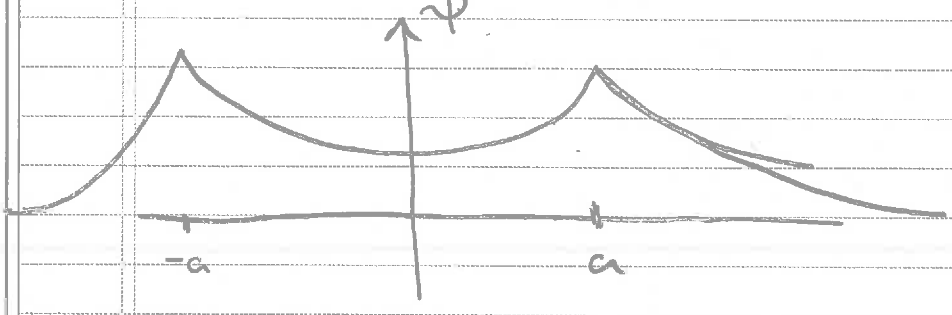



$$K^2 = \frac{-2mE}{\hbar^2} = \frac{z_1^2}{4a^2}$$

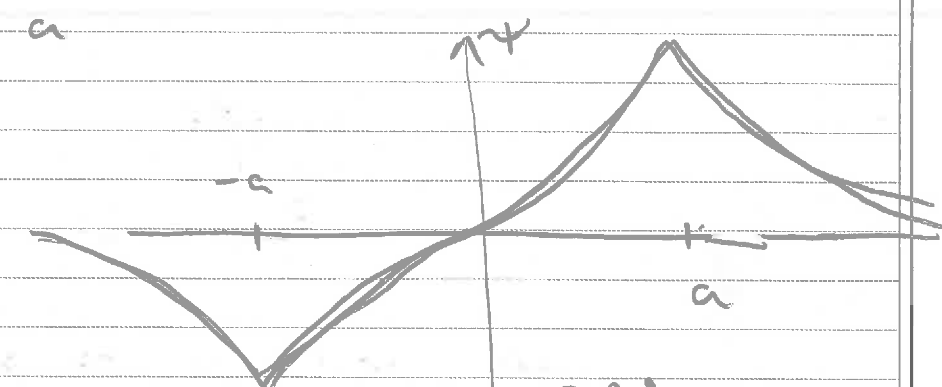
$$E_1 = \frac{-z_1^2}{8}\frac{\hbar^2}{ma^2}$$

$$\boxed{E_1 = -0.317\,\frac{\hbar^2}{ma^2}}$$

$\alpha = \frac{\hbar^2}{4ma} \longrightarrow$ no odd bnd state




Even




Odd