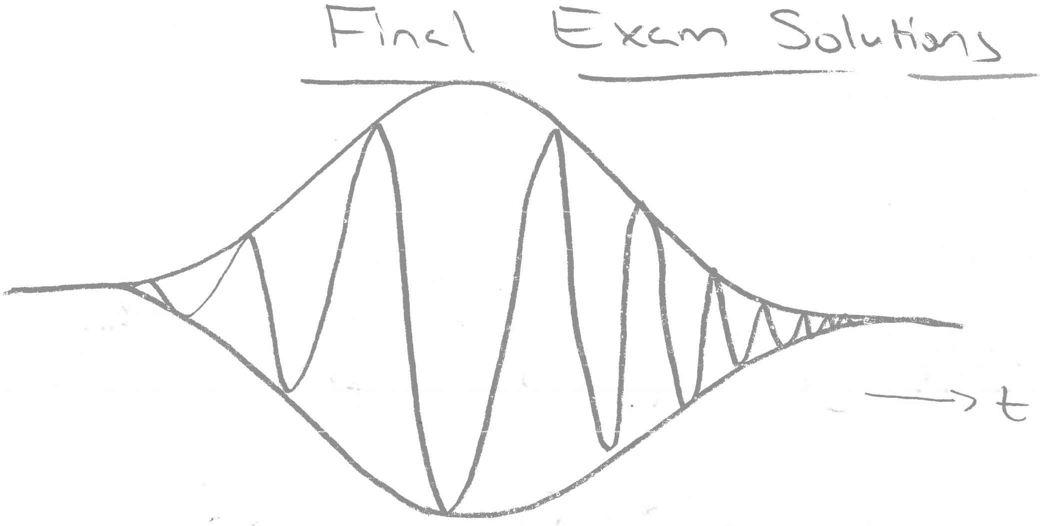
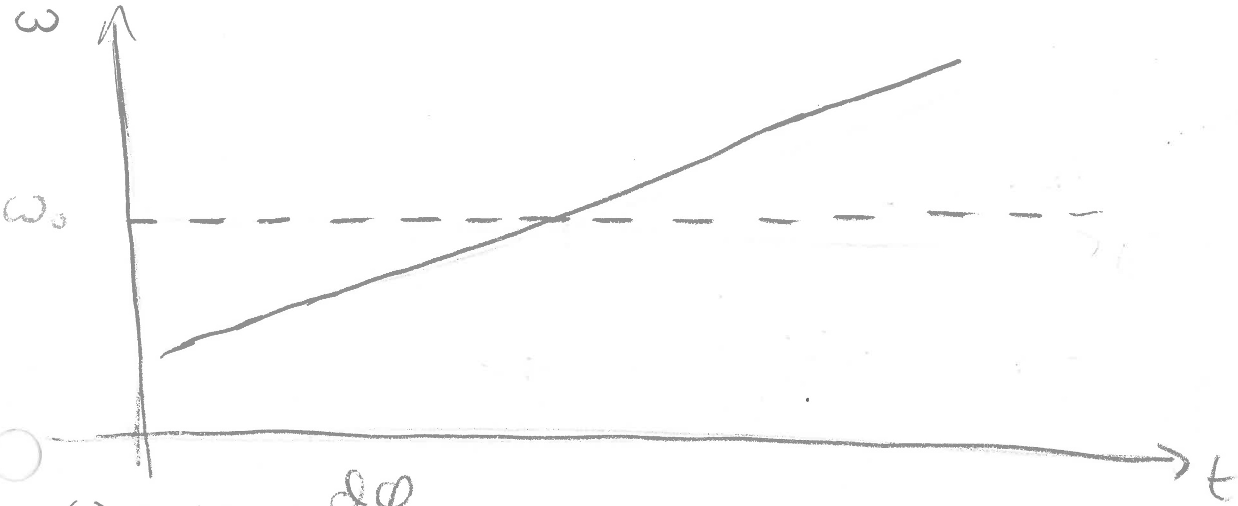
Final Exam Solutions

1.

(a.)





$$\omega = \omega_0 + \frac{d\varphi}{dt}$$

$$\frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2} > 0 \rightarrow \text{up-chirped pulse}$$

(b.)

$$\boxed{\bar{E} = \frac{1}{2} E_0 e^{-\frac{x^2+y^2}{w^2(z)}} e^{-(1+ia)t^2/\tau_0^2} e^{i(\omega_0 t - kz)} e^{i\varphi_0} \hat{x} + c.c.}$$

(c.)

$$I(t) = \frac{1}{2}\epsilon_0 c n \bar{E}^* E = \frac{\epsilon_0 c}{2} E^* E$$

$$E^* E = \left(\frac{1}{2}\right)^2 E_0^2 e^{\frac{-2(x^2+y^2)}{w^2(z)}} \left[e^{-(1-ia)t^2/\tau_0^2} e^{-i(\omega_0 t - kz)} e^{-i\varphi_0} + e^{-(1+ia)t^2/\tau_0^2} e^{i(\omega_0 t - kz)} e^{i\varphi_0} \right]^2$$

$$E^*E = \frac{E_0^2}{4} e^{\frac{-2(x^2+y^2)}{w^2(z)}} \Big[2e^{-2t^2/\tau_G^2} + e^{-2(1-ia)t^2/\tau_G^2} e^{-2i(\omega_0 t - kz)} e^{-2i\varphi_0} +$$

$$e^{-2(1+ia)t^2/\tau_G^2} e^{2i(\omega_0 t - kz)} e^{2i\varphi_0} \Big]$$

$$= \frac{E_0^2}{4} e^{\frac{-2(x^2+y^2)}{w^2(z)}} e^{-2t^2/\tau_G^2} \Big[2 + e^{+2ia t^2/\tau_G^2} e^{+2i(kz - \omega_0 t - \varphi_0)} +$$

$$e^{-2ia\frac{t^2}{\tau_G^2}} e^{-2i(kz - \omega_0 t - \varphi_0)} \Big]$$

$$= \frac{E_0^2}{4} e^{\frac{-2(x^2+y^2)}{w^2(z)}} e^{-2t^2/\tau_G^2} \Big\{ 2\Big[1 + \cos\Big(\frac{2at^2}{\tau_G^2} + kz - \omega_0 t - \varphi_0$$

$$E^*E = \frac{E_0^2}{2} e^{\frac{-2(x^2+y^2)}{w^2(z)}} e^{-2t^2/\tau_G^2} \Big[1 + \cos\Big\{-\Big[\Big(\omega_0 - \frac{2at}{\tau_G^2}\Big)t - kz + \varphi$$

$$\cos(-\theta) = \cos\theta$$

$$\boxed{I(t) = \frac{\epsilon_0 c E_0^2}{4} e^{\frac{-2(x^2+y^2)}{w^2(z)}} e^{-2t^2/\tau_G^2} \Big[1 + \cos\Big[\underbrace{\Big(\omega_0 - \frac{2at}{\tau_G^2}\Big)}_{\omega(t)} t - kz + \varphi_0\Big]\Big]}$$

$$\boxed{\omega = \omega_0 - \frac{2at}{\tau_G^2}}$$

Depending on the sign of a, $\omega(t)$ either increases or decreases w/ t. For our up-chirped pulse, we know that $\omega(t)$ increases w/ t, so a is negative.

2. (a.)

$$I_p = \frac{2\,W}{10^3\,Hz}\,\frac{1}{\pi(10^{-5}m)^2}\,\frac{1}{\tau_G} = \frac{6.37\times10^{6}}{\tau_G}\,\frac{W}{m^2}$$

$$I_p = \frac{cn\epsilon_0}{2}E_p^2 \text{ , where } E_p = 5\times10^{11}\,V/m \text{ , } n=1$$

$$\frac{6.37\times10^{6}\,\frac{W}{m^2}}{\tau_G} = \frac{3\times10^{8}\,m/s \times 8.85\times10^{-12}\,F/m}{2}\,25\times10^{27}\,V^2/m^2$$

$$\boxed{\tau_G = 19.2\text{ fs}}$$

(b.)

$$\Delta\nu_G\tau_G = 0.441 \quad (\text{I'll also accept } \Delta\nu_G\tau_G = \tfrac{1}{2} \text{ and } \Delta\nu_G\tau_G = 1)$$

$$\Delta\nu_G = \frac{0.441}{\tau_G}$$

$$\frac{c}{\lambda} = \nu \longrightarrow \frac{c}{\lambda^2}\Delta\lambda_G = \Delta\nu_G$$

$$\Delta\lambda_G = \frac{\lambda^2}{c}\Delta\nu_G$$

$$\Delta\lambda = \frac{\lambda^2}{c}\,\frac{0.441}{\tau_G}$$

$$= \frac{(8\times10^{-7}m)^2}{3\times10^{8}\,m/s}\,\frac{0.441}{1.92\times10^{-14}s}$$

$$\boxed{\Delta\lambda = 4.90\times10^{-8}m = 49\text{ nm}}$$

$$\boxed{\Delta\lambda = 111\text{ nm},\ \Delta\nu_G\tau_G = 1}$$

$$\boxed{\Delta\lambda = 55.6\text{ nm},\ \Delta\nu_G\tau_G = \tfrac{1}{2}}$$

(c.)

$$\frac{2W}{10^3\,Hz} = 2\times10^{-3}\,J/\text{pulse} = 2\times10^{-3}\,J/\text{pulse} \times \frac{1\,eV}{1.603\times10^{-19}\,J} = 1.25\times10^{16}\,\frac{eV}{\text{pulse}}$$

$$\frac{hc}{\lambda_0} = \frac{1240\text{ nm-eV}}{800\text{ nm}} = 1.55\text{ eV/photon (on average)}$$

$$\frac{1.25\times10^{16}\text{ eV/pulse}}{1.55\text{ eV/photon}} = \boxed{8.05\times10^{15}\text{ photons/pulse}}$$

3.

(a.) $$\Delta\nu_c = \frac{FSR}{\mathcal{F}}$$

$$FSR = \frac{c}{2nL} = \frac{3\times10^8\ m/s}{2\times1.25\times0.40\ m} = 3\times10^8\ Hz$$

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi\sqrt{0.99}}{0.01} = 3.126\times10^2$$

$$\Delta\nu_c = \frac{FSR}{\mathcal{F}} = \frac{1.5\times10^8\ Hz}{3.126\times10^2} = \boxed{9.60\times10^5\ Hz}$$

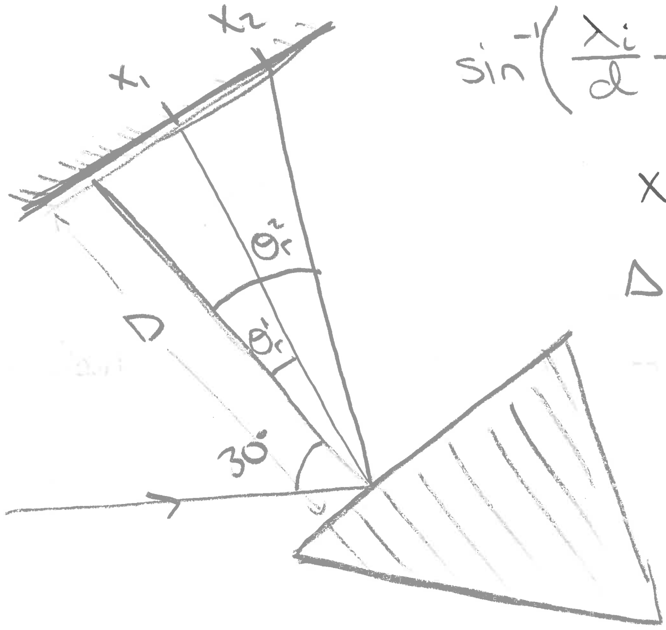
(b.) $$\Delta\nu_L = \frac{(2)\pi h\nu_L\Delta\nu_c^2}{P_L} = \frac{(2)\pi(6.626\times10^{-34}\ J\text{-}s)(4.74\times10^{14}\ Hz)(9.60\times10}{10^{-2}\ W}$$

$$\left[\nu_L = \frac{c}{\lambda} = \frac{3\times10^8\ m/s}{6.328\times10^{-7}\ m} = 4.74\times10^{14}\ Hz\right]$$

$$\Delta\nu_L = 1.82\times10^{-4}\ Hz = \boxed{182\ \mu Hz}$$ if you used a factor of 2

$$\Delta\nu_L = 9.09\times10^{-5}\ Hz = \boxed{90.9\ \mu Hz}$$ if you used a factor of 1

(c.)



$$\sin^{-1}\left(\frac{\lambda_i}{d} - \frac{1}{2}\right) = \theta_r^i$$

$$x_i = D\tan\theta_r^i$$

$$\Delta x = x_2 - x_1 = D(\tan\theta_r^2 - \tan$$

Since $\lambda_0 = 632.8\ nm \Rightarrow \nu_L = 4.74\times10^{14}\ Hz$

and BW of $\Delta\nu_L = 181\ \mu Hz$ is approximately 18 orders of magnitude smaller, we are going to be dealing with a hard to measure linewidth.

$$\Delta x = 20\,\mu m/2 = 10^{-5}\ m$$

$$D = 500\times10^{-3}\ m = 0.5\ m$$

$$\left.\begin{aligned} d &= \frac{10^{-3}}{2000} = 5\times10^{-7}\ m \\ \lambda_1 &= 632.8\ nm \end{aligned}\right] \rightarrow \begin{aligned} \theta_r^1 &= \sin^{-1}\left(\frac{\lambda_1}{d} - \frac{1}{2}\right) \\ \theta_r^1 &= 0.872 \end{aligned}$$

$$\Delta x = D(\tan\theta_r^2 - \tan\theta_r^1)$$

$$\tan^{-1}\left(\frac{\Delta x}{D} + \tan\theta_r^1\right) = \theta_r^2$$

But $\theta_r^2 = \sin^{-1}\left(\frac{\lambda_2}{d} - \frac{1}{2}\right)$

$$\sin^{-1}\left(\frac{\lambda_2}{d} - \frac{1}{2}\right) = \tan^{-1}\left(\frac{\Delta x}{D} + \tan\theta_r^1\right)$$

$$\lambda_2 = d\left[\sin\left(\tan^{-1}\left(\frac{\Delta x}{D} + \tan\theta_r^1\right)\right) + \frac{1}{2}\right]$$

$$= 5\times10^{-7}\ m\left[\sin\left(\tan^{-1}\left[\frac{10^{-5}\ m}{0.15\ m} + \tan(0.872)\right]\right) + \frac{1}{2}\right]$$

$$\lambda_2 = 632.82\ nm$$

If $\Delta\lambda > 2(632.82\ nm - 632.8\ nm)$ or $\boxed{\Delta\lambda > 0.04\ nm}$, then we can resolve it on more than one pixel.

However, $\Delta\nu_L = 181\ \mu Hz$ gives us a bandwidth of $\Delta\lambda_L = \frac{c}{\nu_L^2}\Delta\nu_L$ or $2.4\times10^{-25}\ m$, which is beyond any known measurement technique.

4.

(a.) $\tau_G = 5$ fs , $\lambda_o = 500$ nm , $n = 1$

Method 1

$c\Delta t = \Delta x$

$3\times10^8 \, m/s \times 5\times10^{-15} s = \Delta x$

$3\times5\times10^{-7} m = \Delta x$

$\frac{\Delta x}{\lambda}$ = # of optical cycles in one pulsewidth

$= \frac{3\times\cancel{5}\times10^{-7} m}{\cancel{5}\times10^{-7} m} = \boxed{3.0 \text{ optical cycles}}$ ← standard definition

$\frac{2\Delta x}{\lambda} = \boxed{6.0 \text{ optical cycles in two pulsewidths (also acceptable)}}$

Method 2

$\text{Period} = \frac{1}{\nu} = \frac{\lambda}{c} = \frac{5\times10^{-7} m}{3\times10^8 m/s} = 1.67\times10^{-15} s = 1.67 f$

Pulsewidth = 5 fs

$\frac{\text{Pulsewidth}}{\text{Period}} = \frac{5 fs}{1.67 fs} = \boxed{3.0 \text{ optical cycles}}$

(b.)

For every pulsewidth, τ_G, the carrier is oscillating just 1.5 optical cycles. The temporal portion of our envelope, $e^{-(t/\tau_G)^2}$, changes by e^{-1} (factor of 2.72) in magnitude for every 1.5 carrier oscillations.

SVEA (in temporal form):

$$\left|\frac{d\tilde{E}(t)}{dt}\right| << \omega_o \left|\tilde{E}(t)\right|$$

$$\left|\frac{-t}{\tau_G^2}\cancel{e^{-t^2/\tau_G^2}}\right| \overset{?}{<<} \frac{2\pi c}{\lambda_o}\left|\cancel{e^{-t^2/\tau_G^2}}\right|$$

At $t = \tau_G$

$$\frac{1}{\tau_G} \overset{?}{<<} \frac{2\pi \, 3\times10^8 m/s}{5\times10^{-7} m}$$

$$\frac{1}{5}\cancel{10^{15}} Hz \overset{?}{<<} \pi \cancel{10^{15}} Hz \, \frac{6}{5}$$

$$1 \overset{?}{<<} 6\pi \longrightarrow 1 \lesssim 18$$

Although it is close, we are still in the limit that the envelope is slowly varying with respect to the carrier wave.