SUBJECT

HW6 Answers

DATE e^{-x^2} 1

8.2

$$\psi(x) = e^{if(x)/\hbar}$$

$$\frac{d\psi}{dx} = \frac{i}{\hbar} f'(x) e^{if(x)/\hbar}$$

$$\frac{d^2\psi}{dx^2} = \left(-\frac{[f'(x)]^2}{\hbar^2} + \frac{i}{\hbar} f''(x)\right) e^{if(x)/\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi$$

$$-\frac{f'^2}{\hbar^2} + \frac{i}{\hbar} f'' = -\frac{p^2}{\hbar^2}$$

$$\boxed{i\hbar f'' - (f')^2 + p^2 = 0}$$

(b.) $f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \ldots \hbar^n f_n(x)$

$$f'(x) = f_0'(x) + \hbar f_1'(x) + \hbar^2 f_2'(x) + \ldots$$

$$f''(x) = f_0''(x) + \hbar f_1''(x) + \hbar^2 f_2''(x) + \ldots$$

$$(f_1')^2 = (f_0'(x))^2 + 2\hbar f_0' f_1' + \hbar^2(2f_0'f_2' + f_1'^2) + 2\hbar^3[f_0'f_3' + f_1'f_2'] + O(\hbar^4)$$

$$i\hbar f'' = i(\hbar f_0'' + \hbar^2 f_1'' + \hbar^3 f_2'' + O(\hbar^4))$$

$$(i\hbar f_0'' + i\hbar^2 f_1'' + i\hbar^3 f_2'') - [f_0'^2 + 2\hbar f_0' f_1' + \hbar^2(2f_0'f_2' + f_1'^2) + 2\hbar^3[f_0'f_3' + f_1'f_2']] + p^2 = 0$$

Grouping like powers of $\hbar$:

$\hbar^0$: $-f_0'^2 + p^2 = 0 \rightarrow (f_0')^2 = p^2$

$\hbar^1$: $if_0'' - 2f_0'f_1' = 0 \rightarrow if_0'' = 2f_0'f_1'$

$\hbar^2$: $if_1'' - 2f_0'f_2' - (f_1')^2 = 0 \rightarrow if_1'' = 2f_0'f_2' + (f_1')^2$

$\hbar^3$: $if_2'' - 2(f_0'f_3' + f_1'f_2') = 0 \rightarrow if_2'' = 2f_0'f_3' + 2f_1'f_2'$

(c.) $(f_0')^2 = p^2 \rightarrow f_0' = \pm p$

$f_0 = \pm\int p(x)dx + C$ ②

$i f_0'' = 2 f_0' f_1'$

$\frac{i}{2}\frac{f_0''}{f_0'} = f_1' \rightarrow f_1' = \frac{i}{2}\left(\frac{\pm p'}{\pm p}\right) = \frac{i}{2}\left(\frac{\frac{dp}{dx}}{p}\right)$

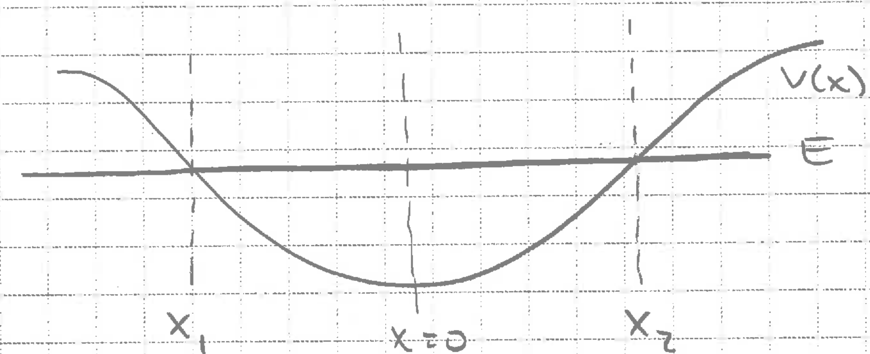
$f_1' = \frac{i}{2}\frac{d}{dx}\left(\frac{dp}{p}\right) \Rightarrow \frac{i}{2}\frac{d}{dx}[\ln(p)] \Rightarrow f_1 = \frac{i}{2}\ln p + C$

$\psi = \exp\left(if(x)/\hbar\right) = \exp\left[\frac{i}{\hbar}\left(\underbrace{\pm\int p(x)dx}_{f_0} + \underbrace{\frac{i\hbar}{2}\ln(p)}_{f_1} + C_0\right)\right]$

$\psi = \exp\left(\frac{\pm i}{\hbar}\int p(x)dx\right)\exp\left(-\frac{1}{2}\ln(p)\right)C$

$$\boxed{\psi = \frac{C}{\sqrt{p}}\exp\left[\pm\frac{i}{\hbar}\int p(x)dx\right] \quad \text{QED}}$$

8.7



We need to use the potential well w/ no walls:

$$\int_{x_1}^{x_2} p(x)dx = \pi\hbar\left(n-\tfrac{1}{2}\right) \quad \text{Eqn. 8.51}$$

$p(x) = \sqrt{2m\left(E - \frac{1}{2}m\omega^2x^2\right)}$

$E = \frac{1}{2}m\omega^2x_1^2$, where $x_1 = -x_2$

$-\sqrt{\frac{2E}{m\omega^2}} = x_1$

$\sqrt{\frac{2E}{m\omega^2}} = x_2$

$$(n-\tfrac{1}{2})\pi\hbar = \int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2} \sqrt{2m}\left[\left(\frac{2E}{m\omega^2} - x^2\right)\frac{1}{2}m\omega^2\right]^{1/2} dx$$

$$= 2m\omega \int_0^{x_2} \sqrt{x_2^2 - x^2}\,dx \quad \text{, where } x_2 \equiv \sqrt{\frac{2E}{m\omega^2}}$$

$$= m\omega\left(x\sqrt{x_2^2 - x^2} + x_2^2 \sin^{-1}\left(\frac{x}{x_2}\right)\right)\Bigg|_0^{x_2} = m\omega\left[0 + x_2^2\sin^{-1}(1) + \cancel{x_2^2\sin^{-1}}(0)\right]$$

$$\sin^{-1}(1) = \frac{\pi}{2} \qquad \sin^{-1}(0) = 0$$

$$(n-\tfrac{1}{2})\cancel{\pi}\hbar = m\omega x_2^2 \frac{\cancel{\pi}}{2}$$

$$= \frac{1}{\cancel{2}}\cancel{m}\omega\frac{\cancel{2}E}{\cancel{m}\omega^{\cancel{2}}} = \frac{E}{\omega}$$

$\hbar\omega(n-\frac{1}{2}) = E$ for $n = 1, 2, 3, \dots$

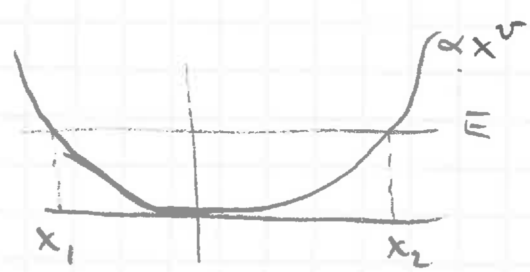
But since $n-\frac{1}{2}$ for $n=1$ is $\frac{1}{2}$, we can switch to $E = \hbar\omega(n+\frac{1}{2})$ for $n = 0, 1, 2, \dots$ instead, which is exactly the result for the HO.

8.11 Again, we need to use Eqn. 8.51

$$(n-\tfrac{1}{2})\pi\hbar = \int_{x_1}^{x_2} p(x)\,dx$$

Since we're dealing w/ an abs. value in our potential, we know that $p(x)$ is even so:

$$(n-\tfrac{1}{2})\pi\hbar = 2\int_0^{x_2} p(x)\,dx$$



$$p(x) = \sqrt{2m(E - \alpha x^\nu)}$$

$$E = \alpha x_2^\nu \qquad z \equiv \frac{\alpha}{E}x^\nu$$

$$x = \left(\frac{zE}{\alpha}\right)^{1/\nu} \qquad dx = \left(\frac{E}{\alpha}\right)^{1/\nu} \frac{1}{\nu} z^{\frac{1}{\nu}-1}\, dz$$

$$\left(n - \tfrac{1}{2}\right)\pi\hbar = 2\sqrt{2mE}\int_0^{x_2} \left(1 - \frac{\alpha}{E} x^{\nu}\right)^{1/2} dx$$

$$= 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu} \frac{1}{\nu}\int_0^1 z^{1/\nu - 1}\sqrt{1-z}\, dz = 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu} \frac{\Gamma(1/\nu)\Gamma(3/2)}{\Gamma\left(\frac{1}{\nu}+3/2\right)}$$

$$= 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu} \frac{\Gamma(1/\nu+1)\frac{1}{2}\sqrt{\pi}}{\Gamma(1/\nu+3/2)} = \sqrt{2\pi mE}\left(\frac{E}{\alpha}\right)^{1/\nu} \frac{\Gamma(1/\nu+1)}{\Gamma(1/\nu+3/2)}$$

$$(n-1/2)\pi\hbar = \sqrt{2\pi m}\left(\frac{1}{\alpha}\right)^{1/\nu} \frac{\Gamma(1/\nu+1)}{\Gamma(1/\nu+3/2)} E^{1/\nu+1/2}$$

$$\frac{(n-1/2)\pi\hbar}{\sqrt{2\pi m}}\alpha^{1/\nu} \frac{\Gamma(1/\nu+3/2)}{\Gamma(1/\nu+1)} = E^{1/\nu+1/2}$$

$$\therefore\ \boxed{E_n = \alpha\left[(n-1/2)\hbar\sqrt{\frac{\pi}{2m\alpha}}\ \frac{\Gamma(1/\nu+3/2)}{\Gamma(1/\nu+1)}\right]^{\frac{2\nu}{\nu+2}}}$$

$\nu = 2$

$$E_n = \alpha(n-1/2)\hbar\sqrt{\frac{\pi}{2m\alpha}}\ \frac{\Gamma(2)}{\Gamma(3/2)}$$

$1 \Rightarrow \Gamma(n) = (n-1)!$

$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma(3/2) = 1/2\, \Gamma(1/2)$

$\Gamma(1/2) \equiv \sqrt{\pi}$ so:

$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$

$$= \cancel{\alpha(n-1/2)\hbar\frac{2}{\sqrt{\pi}}\sqrt{\frac{\pi}{2m\alpha}}}$$

$$\boxed{E_n = (n-1/2)\hbar\sqrt{\frac{2\alpha}{m}}}$$

For the HO, $\alpha = \frac{1}{2}m\omega^2$, so: $E_n = (n-1/2)\hbar\sqrt{\frac{2(\frac{1}{2}m\omega^2)}{m}}$

$$\boxed{E_n = (n-1/2)\hbar\omega}$$

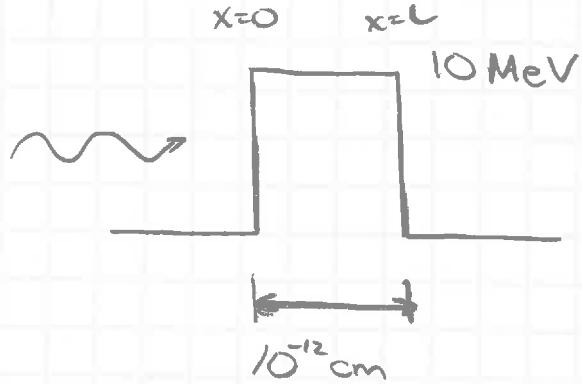
✓

4.

$KE = 4\ \text{MeV}$

$L = 10^{-12}\ \text{cm}$

$V_0 = 10\ \text{MeV}$



$$\psi_I(x) = A_0 e^{ikx} + A e^{-ikx} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi_{II}(x) = B e^{-\alpha x} + C e^{+\alpha x} \qquad \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$x = 0$

① $A_0 + A = B + C$ (continuity of $\psi(x)$)

② $ik(A_0 - A) = -\alpha(B - C)$ (continuity of $\psi'(x)$)

$x = L$

③ $De^{ikL} = Be^{-\alpha L} + Ce^{+\alpha L}$ (continuity of $\psi(x)$)

④ $ikDe^{ikL} = -\alpha(Be^{-\alpha L} - Ce^{+\alpha L})$ (continuity of $\psi'(x)$)

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① $A_0 + A = B + C$

$\oplus$

② $A_0 - A = \frac{-\alpha}{ik}(B - C)$

$$2A_0 = B\left(1 - \frac{\alpha}{ik}\right) + C\left(1 + \frac{\alpha}{ik}\right)$$

$$\boxed{2ikA_0 = B(ik - \alpha) + C(ik + \alpha)}$$

③ $De^{ikL} - Ce^{+\alpha L} = Be^{-\alpha L}$

④ $\frac{ik}{-\alpha} De^{ikL} = Be^{-\alpha L} - Ce^{+\alpha L}$

Putting ③ into ④

$$-\frac{ik}{\alpha} De^{ikL} = De^{ikL} - Ce^{+\alpha L} - Ce^{+\alpha L}$$

$$-\left(1+\frac{ik}{\alpha}\right)De^{ikL} = -2Ce^{+\alpha L}$$

$$\frac{1}{2\alpha}(\alpha+ik)De^{ikL} = Ce^{+\alpha L}$$

$$\frac{1}{2\alpha}(\alpha+ik)De^{(ik-\alpha)L} = C$$

3 $$De^{ikL} - Ce^{+\alpha L} = Be^{-\alpha L}$$

$$De^{ikL}\left(1-\frac{1}{2\alpha}(\alpha+ik)\right) = Be^{-\alpha L}$$

$$De^{(ik+\alpha)L}\left[1-\frac{1}{2\alpha}(\alpha+ik)\right] = B$$

$$2ikA_0 = B(ik-\alpha) + C(ik+\alpha)$$

$$2ikA_0 = De^{ikL}\left\{\left[e^{\alpha L}\left[1-\frac{1}{2\alpha}(\alpha+ik)\right]\right](ik-\alpha) + \left[\frac{1}{2\alpha}(\alpha+ik)^2\right]e^{-\alpha L}\right\}$$

$$4i\alpha k A_0 = De^{ikL}\left\{-(\alpha-ik)^2 e^{\alpha L} + (\alpha+ik)^2 e^{-\alpha L}\right\}$$

$$\frac{D}{A_0} = \frac{4i\alpha k e^{-ikL}}{-(\alpha-ik)^2 e^{\alpha L} + (\alpha+ik)^2 e^{-\alpha L}}$$

Let's compute αL first: $\alpha L = \frac{\sqrt{2m(V_0-E)}}{\hbar}L$

$$\alpha_p L = 10^{-14}\text{m}\,\frac{\sqrt{2\times1.67\times10^{-27}\text{ kg}\,(6\text{ MeV})\times1.602\times10^{-13}\text{J}}}{1.055\times10^{-34}\text{ J}\cdot\text{s}}$$

$m_D = 3.34\times10^{-27}$ kg

$m_p = 1.67\times10^{-27}$ kg

$\alpha_p L = 5.37$ (bigger for deutron)

If $\alpha L \gg 1$, then:

$$\frac{D}{A_0} \simeq \frac{4i\alpha k e^{-ikL}}{-(\alpha-ik)^2 e^{\alpha L}} \quad \text{so} \quad T = \left|\frac{D}{A_0}\right|^2 \simeq \frac{16\alpha^2k^2}{(\alpha^2+k^2)^2}e^{-2\alpha L}$$

$$T_p = \frac{16\alpha_p^2 k_p^2}{(\alpha_p^2+k_p^2)^2}e^{-2(5.37)}$$

$$\alpha_p = \frac{\sqrt{2m_p(V_0-E)}}{\hbar} = 5.37\times10^{+14}\text{ cm}^{-1}$$

$$k_p = \frac{\sqrt{2m_pE}}{\hbar} = 4.39\times10^{+14}\text{ cm}^{-1}$$

$$T_p \simeq \frac{16\,(5.37\times10^{+14}\ cm^{-1})^2(4.39\times10^{+14}\ cm^{-1})^2}{\underbrace{\left[(5.37\times10^{14}\ cm^{-1})^2+(4.39\times10^{14}\ cm^{-1})^2\right]^2}_{3.84}}\,e^{-10.74}$$

$$\boxed{T_p \cong 8.32\times10^{-5}}$$

$$T_D \simeq \frac{16\alpha_D^2 k_D^2}{(\alpha_D^2+k_D^2)^2}e^{-2\alpha L}$$

$$\alpha_D = \frac{\sqrt{2m_D(V_0-E)}}{\hbar} = 7.61\times10^{14}\ cm^{-1}$$

$$k_D = \frac{\sqrt{2m_D E}}{\hbar} = 6.22\times10^{14}\ cm^{-1}$$

$$T_D = \frac{16(7.61\times10^{14}\ cm^{-1})^2(6.22\times10^{14}\ cm^{-1})^2}{\underbrace{\left[(7.61\times10^{14}\ cm^{-1})^2+(6.22\times10^{14}\ cm^{-1})^2\right]}_{3.84}}\,e^{-2(7.61)} = 9.4\times10^{-7}\simeq10^{-6}$$

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5. (a.) $V(x) = mgh = mgx$ for $x>0$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx}+mgx\psi = E\psi$$

$$\frac{d^2\psi}{dx} = -2m\left(\frac{E-mgx}{\hbar^2}\right)\psi$$

$$\frac{d^2\psi}{dx^2} = \underbrace{\frac{2m^2g}{\hbar^2}}_{\alpha^3}\underbrace{\left(x-\frac{E}{mg}\right)}_{y}\psi$$

$$\frac{d^2\psi}{dy^2} = \alpha^3 y\psi$$

We can simplify this further (and put it into the Airy eqn form) by choosing $z=\alpha y$

$$\cancel{\alpha^2}\frac{d^2\psi}{dz^2} = \cancel{\alpha^2}z\psi$$

$$\frac{d^2\psi}{dz^2} = z\psi \quad \text{(Airy eqn.)}$$

The general soln is: $\psi = aAi(z)+bBi(z)$

Since $Bi(z)$ blows up for large z, $b = 0$.

$$\boxed{\therefore \quad \psi(x) = a\,Ai(z) = a\,Ai\left[\alpha\left(x - \frac{E}{mg}\right)\right], \text{ where } \alpha = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}}$$

(b.) $\psi(0) = 0$ since $V(0) = \infty$

$\therefore$ At the zeroes of $Ai(z)$ we can find the allowed energies. $Ai(z) = 0$ @ $a_1 = -2.34$, $a_2 = -4.09$, and $a_3 = -5.52$.

① $$\alpha\left(-\frac{E_1}{mg}\right) = -2.34 \rightarrow \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \frac{E_1}{mg} = 2.34$$

$$E_1 = \left(\frac{\hbar^2 m g^2}{2}\right)^{1/3} 2.3,$$

$m = 0.1$ kg

$g = 9.8\ m^2/s$

$$\boxed{E_2 = 8.81 \times 10^{-23}\ J}$$

② $$E_2 = \left(\frac{\hbar^2 m g^2}{2}\right)^{1/3} 4.09$$

$$\boxed{E_2 = 1.54 \times 10^{-22}\ J}$$

③ $$E_3 = \left(\frac{\hbar^2 m g^2}{2}\right)^{1/3} 5.52$$

$$\boxed{E_3 = 2.08 \times 10^{-22}\ J}$$

(c.) $$\boxed{E_1 = \left(\frac{\hbar^2 m_e g^2}{2}\right)^{1/3} 2.34 = 1.15 \times 10^{-13}\ eV}$$

$$2\langle T \rangle = \left\langle x \frac{dV}{dx} \right\rangle \qquad \frac{dV}{dx} = \frac{d}{dx}(mgx) = mg$$

$$\left\langle x \frac{dV}{dx} \right\rangle = \langle mgx \rangle = \langle V \rangle$$

$$2\langle T \rangle = \left\langle x \frac{dV}{dx} \right\rangle = \langle V \rangle \rightarrow \langle T \rangle = \frac{1}{2}\langle V \rangle$$

$$\langle H \rangle = \langle T \rangle + \langle V \rangle = E_n$$

$$= \tfrac{1}{2}\langle V \rangle + \langle V \rangle = E_n$$

$$= \tfrac{3}{2}\langle V \rangle = E_n$$

$$= \tfrac{3}{2} mg \langle x \rangle = E_n$$

$$\langle x \rangle = \frac{2E_n}{3mg}$$

$$\langle x \rangle = \frac{2E_1}{3mg} = \frac{2(1.15\times10^{-13}\,eV \times 1.602\times10^{-19}\,J/eV)}{3(9.11\times10^{-31}\,kg)(9.8\,m^2/s)} = 1.3747\times10^{-3}\,m$$

$$\boxed{\langle x \rangle = 1.37\ mm}$$