HW Set 1

1. $$\Phi(\bar{p}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int e^{-i(\bar{p}\cdot\bar{r})/\hbar}\, \psi(\bar{r})\, d^3\bar{r}$$

$$\psi_{100} = \sqrt{\frac{1}{\pi a^3}}\, e^{-r/a}$$, where a = Bohr radius

$$\Phi(\bar{p}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \sqrt{\frac{1}{\pi a^3}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-r/a}\, e^{-i(pr\cos\theta)/\hbar}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$\cos\theta = x$

$-\sin\theta\, d\theta = dx$

$$\phi(\bar{p}) = \frac{-\cancel{2\pi}}{\cancel{2}\pi^2\hbar}\sqrt{\frac{1}{2ha^3}} \int_{+1}^{-1}\int_0^{\infty} e^{-r/a}\, e^{-ipxr/\hbar}\, r^2\, dx\, dr$$

$$= \frac{-1}{a\pi\hbar\sqrt{2a\hbar}} \int_0^{\infty} e^{-r/a}\, r^2 \left[\frac{\hbar}{-ipr}\left(e^{+ipr/\hbar} - e^{-ipr/\hbar}\right)\right] dr$$

$$= \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[\int_0^{\infty} r e^{-r/a}\, e^{ipr/\hbar}\, dr - \int_0^{\infty} r e^{-r/a}\, e^{-ipr/\hbar}\, dr\right]$$

$$= \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[\int_0^{\infty} r e^{-\frac{r}{a}\left(1-\frac{ipa}{\hbar}\right)}\, dr - \int_0^{\infty} r e^{-\frac{r}{a}\left(1+\frac{ipa}{\hbar}\right)}\, dr\right]$$

$$\alpha = \frac{1}{a}\left(1-\frac{ipa}{\hbar}\right) \qquad \beta = \frac{1}{a}\left(1+\frac{ipa}{\hbar}\right)$$

$$= \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[\int_0^{\infty} r e^{-\alpha r}\, dr - \int_0^{\infty} r e^{-\beta r}\, dr\right]$$

$$\int r e^{-\alpha r}\,dr \longrightarrow \begin{array}{ccc} r & \oplus & e^{-\alpha r} \\ 1 & \ominus & -\frac{1}{\alpha} e^{-\alpha r} \\ 0 & & \frac{1}{\alpha^2} e^{-\alpha r} \end{array} \longrightarrow \left. -\frac{r}{\alpha} e^{-\alpha r} - \frac{1}{\alpha^2} e^{-\alpha r} \right|_0^\infty$$

$$\phi(p) = \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[-\frac{r}{\alpha} e^{-\alpha r} - \frac{1}{\alpha^2} e^{-\alpha r} + \frac{r}{\beta} e^{-\beta r} + \frac{1}{\beta^2} e^{-\beta r} \right]_0^\infty$$

$$= \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right] = \frac{1}{i\pi a p\sqrt{2a\hbar}} \left[\frac{\beta^2 - \alpha^2}{\alpha^2 \beta^2} \right]$$

$$\beta^2 - \alpha^2 = \frac{1}{a^2}\left[1 + \frac{2ipa}{\hbar} - \frac{p^2a^2}{\hbar^2} - 1 + \frac{2ipa}{\hbar} + \frac{p^2a^2}{\hbar^2} \right] = \frac{4ip}{a\hbar}$$

$$\alpha^2\beta^2 = \frac{1}{a^4}\left[\left(1 - \frac{ipa}{\hbar}\right)\left(1 + \frac{ipa}{\hbar}\right) \right]^2 = \frac{1}{a^4}\left(1 + \frac{p^2a^2}{\hbar^2}\right)^2$$

$$\boxed{\phi(p) = \frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2} \left[\frac{1}{1 + \left(\frac{pa}{\hbar}\right)^2} \right]^2}$$

(b.)

$$\int_{V_p} \Phi^*(\bar{p})\,\Phi(\bar{p})\,dV_p \stackrel{?}{=} 1$$

$$\frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left[\frac{1}{1 + \left(\frac{pa}{\hbar}\right)^2} \right]^4 p^2 \sin\theta \, dp\, d\theta\, d\varphi \stackrel{?}{=} 1$$

$$\frac{4}{\pi}\left(\frac{2a}{\hbar}\right)^3 \int_0^{\infty} \frac{p^2}{\left[1 + \left(\frac{pa}{\hbar}\right)^2\right]^4}\, dp \stackrel{?}{=} 1$$

$$\int \frac{y^2}{(b^2+y^2)^4}\, dy = \frac{\pi}{32} b^{-5/2}$$

$$\frac{4}{\pi}\left(\frac{2a}{\hbar}\right)^3 \left[\left(\frac{\hbar}{a}\right)^8 \frac{\pi}{32}\left(\frac{\hbar}{a}\right)^{-5}\right] \stackrel{?}{=} 1$$

$$\frac{\cancel{8}a^3}{\hbar^3}\left(\frac{\hbar}{a}\right)^3 \frac{1}{\cancel{8}} \stackrel{?}{=} 1$$

$$1 = 1 \checkmark$$

A plot of $\Psi(\bar{r})$ vs $\bar{r}$ and $\phi(\bar{p})$ vs $\bar{p}$ is attached.

(c.) $$\langle r^2 \rangle = \int \psi^*_{100}\, r^2 \psi_{100}\, dV$$

$$= \frac{1}{\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^4 e^{-2r/a} \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{4}{a^3}\underbrace{\int_0^{\infty} r^4 e^{-2r/a}\, dr}$$

r^4	$+$	$e^{-2r/a}$
$4r^3$	$-$	$-\frac{a}{2}e^{-2r/a}$
$12r^2$	$+$	$\left(\frac{a}{2}\right)^2 e^{-2r/a}$
$24r$	$-$	$-\left(\frac{a}{2}\right)^3 e^{-2r/a}$
24	$+$	$(a/2)^4 e^{-2r/a}$
0		$-(a/2)^5 e^{-2r/a}$

Because we are integrating from 0 to ∞, only the last term survives:

$$\int_0^{\infty} r^4 e^{-2r/a}\, dr = 24\left(\frac{a}{2}\right)^5$$

$$\langle r^2 \rangle = \frac{4}{a^3} 24 \left(\frac{a}{2}\right)^5 = 3a^2$$

$$\langle p^2\rangle = \frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{p^4 \sin\theta}{\left[1+\left(\frac{pa}{\hbar}\right)^2\right]^4} dp\, d\varphi\, d\theta$$

$$= \frac{4}{\pi}\left(\frac{2a}{\hbar}\right)^3 \int_0^{\infty} \frac{p^4}{\left[1+\left(\frac{pa}{\hbar}\right)^2\right]^4} dp$$

$$\int \frac{y^4\, dy}{[b+y^2]^4} = \left(\frac{\pi}{32}\right) m^{-3/2} \quad \text{(from integral table)}$$

$$\langle p^2\rangle = \frac{4}{\pi}\left(\frac{2a}{\hbar}\right)^3 \left(\frac{\hbar}{a}\right)^8 \left(\frac{\pi}{32}\right)\left(\frac{\hbar}{a}\right)^{-3} = \left(\frac{\hbar}{a}\right)^2$$

Whereas $\langle r^2\rangle \propto a^2$, $\langle p^2\rangle$ goes as $\frac{1}{a^2}$.

(d.) $$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{2m}\frac{\hbar^2}{a^2}$$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$\langle T\rangle = \frac{\hbar^2}{2m}\frac{(me^2)^2}{(4\pi\epsilon_0\hbar^2)^2} = \frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2} = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -E_1$$

$$\langle T\rangle = -E_1 = 13.6\,\text{eV}$$

2.

(a.)

(1)

$$E^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \pm i \\ 1 \end{bmatrix}$$

Alternatively, we can write $\frac{1}{2}\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$ since both are equivalent descriptions of circularly polarized light.

(2)

$$E^{(2)} = \frac{1}{2}\begin{bmatrix} \mp i \\ 1 \end{bmatrix} \text{ or } \frac{1}{2}\begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

A mirror transforms right-circularly polarized light into left-circularly polarized and vice versa.

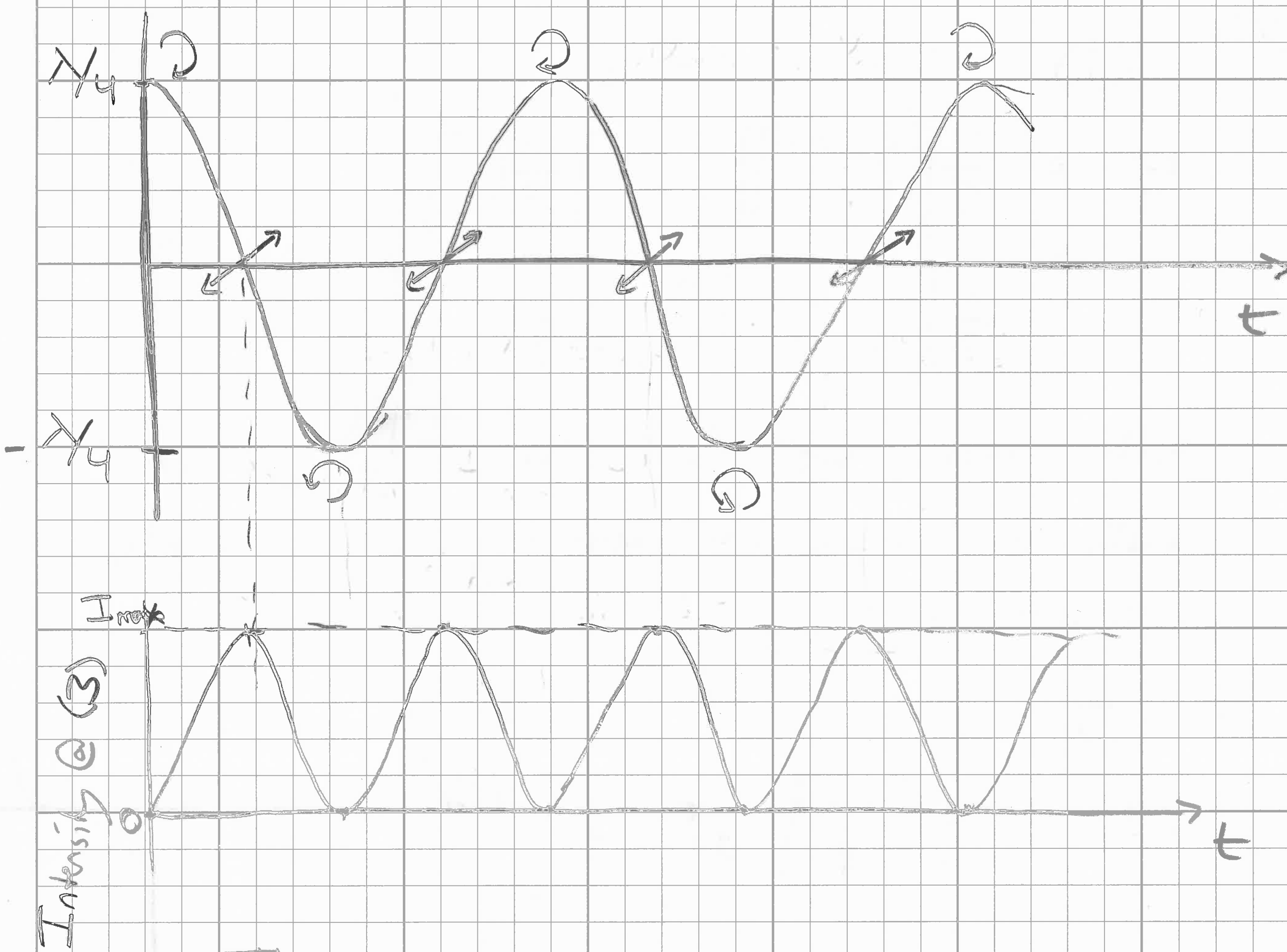
(3)

$$E^{(3)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If and only if we have circularly polarized light leaving the QWP do we have a null at our detector. This is the principle of a <u>Faraday</u> <u>isolator</u>.

(6.)



The intensity is modulated @ twice the frequency of the PEM, w/ zeroes @ ±λ/4 retardation and maximums @ 0 retardation.

3. (a.)

$$\bar{\mu}_m(t) = \int_V \bar{r} \times \bar{J}_e(\bar{r},t)\,dV$$

$$\bar{J}_e(t) = -\frac{ei\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$$

$$\psi_{211} = \sqrt{\frac{1}{24}}\, a^{-3/2}\,\frac{r}{a}\, e^{-r/2a}\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{i\phi}$$

$$\psi_{211} = -\frac{r}{8a^2\sqrt{a\pi}}\sin\theta\, e^{-r/2a+i\phi} = \frac{-1}{8a^2\sqrt{a\pi}}\left[r\sin\theta\, e^{-r/2a+i\phi}\right]$$

$$\psi^*_{211} = \frac{-r}{8a^2\sqrt{a\pi}}\sin\theta\, e^{-r/2a-i\phi} = \frac{-1}{8a^2\sqrt{a\pi}}\left[r\sin\theta\, e^{-r/2a-i\phi}\right]$$

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$$

$$\nabla\psi^* = \frac{-1}{8a^2\sqrt{a\pi}}\left[\sin\theta\, e^{-i\phi}\left(e^{-r/2a} - \frac{r}{2a}e^{-r/2a}\right)\hat{r} + e^{-r/2a-i\phi}\cos\theta\,\hat{\theta} - i e^{-r/2a-i\phi}\hat{\phi}\right]$$

$$\nabla\psi = \frac{-1}{8a^2\sqrt{a\pi}}\left[\sin\theta\, e^{i\phi}\left(e^{-r/2a} - \frac{r}{2a}e^{-r/2a}\right)\hat{r} + e^{-r/2a+i\phi}\cos\theta\,\hat{\theta} + i e^{-r/2a+i\phi}\hat{\phi}\right]$$

$$\bar{J}_e(t) = \frac{-ei\hbar}{2m}\frac{1}{64a^5\pi}\left\{e^{-r/a}\, r\sin\theta\left[\sin\theta\left(1-\frac{r}{2a}\right)\hat{r} + \cos\theta\,\hat{\theta} - i\hat{\phi}\right] - e^{-r/a}\, r\sin\theta\left[\sin\theta\left(1-\frac{r}{2a}\right)\hat{r} + \cos\theta\,\hat{\theta} + i\hat{\phi}\right]\right\}$$

All terms, except in the $\hat{\phi}$ direction, cancel

$$\bar{J}_e(t) = \frac{-e\hbar}{64ma^5\pi}\, e^{-r/a}\, r\sin\theta\,\hat{\phi}$$

$$\bar{\mu}_m(t) = \int_V \bar{r} \times \bar{J}_e(\bar{r}, t)\, dV$$

$$\bar{r} = r\hat{r} \quad \longrightarrow \quad \hat{r} \times \hat{\phi} = -\hat{\theta}$$

$$= \frac{+e\hbar\hat{\theta}}{64 m a^5 \pi} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 \sin\theta\, e^{-r/a}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{+e\hbar\hat{\theta}}{32 m a^5} \int_0^{\pi}\int_0^{\infty} r^4 \sin^2\theta\, e^{-r/a}\, dr\, d\theta$$

$$\int_0^{\pi} \sin^2\theta\, d\theta = \int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \pi/2$$

$$\int_0^{\infty} r^4 e^{-r/a}\, dr \longrightarrow$$

r^4		$e^{-r/a}$
$4r^3$	$\oplus$	$-a\, e^{-r/a}$
$12r^2$	$\ominus$	a^2 "
$24r$	$\oplus$	$-a^3$ "
24	$\ominus$	a^4 "
0	$\oplus$	$-a^5$ "

Since we are integrating from 0 to ∞, only the last term survives, so the integral is equal to $24a^5$

$$\therefore \mu_m(t) = \frac{+e\hbar\hat{\theta}}{\cancel{32}_{8}\, m \cancel{a^5}} \frac{\pi}{\cancel{2}} \cancel{24}^{\cancel{12}\,3} \cancel{a^5} = \boxed{+\frac{3e\hbar\pi}{8m}\hat{\theta}}$$

(b.)

$$\langle \mu_m(t) \rangle = \langle \Psi_{211} | \mu_m(t) | \Psi_{211} \rangle$$

Since $\mu_m(t)$ doesn't Δ $|\Psi_{211}\rangle$, we can pull it out:

$$\langle \mu_m(t) \rangle = \mu_m(t) \langle \Psi_{211} | \Psi_{211} \rangle$$

Since $\langle \Psi_{n\ell m} | \Psi_{n\ell m} \rangle = 1$, we have

$$\langle \mu_m(t) \rangle = \mu_m(t) = \boxed{\frac{3e\hbar\pi}{8m}\,\hat{\theta}}$$

4. $|\psi_I\rangle = e^{iH_0t/\hbar}|\psi_s\rangle$

$$\frac{d}{dt}|\psi_I\rangle = iH_0/\hbar\, e^{iH_0t/\hbar}|\psi_s\rangle + e^{iH_0t/\hbar}\frac{\partial}{\partial t}|\psi_s\rangle$$

Using the SE: $i\hbar\frac{\partial}{\partial t}|\psi_s\rangle = H|\psi_s\rangle$, we can substitute in $\frac{1}{i\hbar}H|\psi_s\rangle$ for $\frac{\partial}{\partial t}|\psi_s\rangle$ to get

$$\frac{d}{dt}|\psi_I\rangle = \frac{-H_0}{i\hbar}|\psi_I\rangle + e^{iH_0t/\hbar}\frac{1}{i\hbar}H|\psi_s\rangle$$

$$= \frac{-H_0}{i\hbar}|\psi_I\rangle + \frac{1}{i\hbar}\underbrace{(H_0 + H_I')}_{H}|\psi_I\rangle$$

$$\frac{d}{dt}|\psi_I\rangle = \frac{1}{i\hbar}H_I'|\psi_I\rangle \qquad \text{QED}$$

$$Q_I(t) = e^{iH_0t/\hbar}\hat{Q}_s e^{-iH_0t/\hbar}$$

$$\frac{d}{dt}Q_I(t) = iH_0/\hbar\, \underbrace{e^{iH_0t/\hbar}\hat{Q}_s e^{-iH_0t/\hbar}}_{\hat{Q}_I} + e^{iH_0t/\hbar}\frac{\partial\hat{Q}_s}{\partial t}e^{-iH_0t/\hbar} - iH_0t/\hbar\, \underbrace{e^{iH_0t/\hbar}\hat{Q}_s e^{-iH_0t/\hbar}}_{\hat{Q}_I}$$

$$= -\frac{1}{i\hbar}H_0\hat{Q}_I + \frac{1}{i\hbar}\hat{Q}_I\hat{H}_0 + \frac{\partial Q_I}{\partial t}$$

$$\frac{d}{dt}Q_I = \frac{1}{i\hbar}[\hat{Q}_I, \hat{H}_0] + \frac{\partial Q_I}{\partial t} \qquad \text{QED}$$

94

94