① (22 points)

$$\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda \longrightarrow \begin{cases} \nu\lambda = c \\ \nu = c/\lambda \\ \delta\nu = \frac{c}{\lambda^2}\delta\lambda \end{cases}$$

$$\frac{1240\ \text{nm-eV}}{1.55\ \text{eV}} = \frac{hc}{\nu} = \lambda$$

$$800\ \text{nm} = \lambda$$

$$\Delta\nu = \frac{3\times10^8\ \text{m/s}}{(8\times10^{-7}\ \text{m})^2}\ 2.14\ \text{nm} = 10^{12}\ \text{Hz}$$

$$\alpha_{max} L = 0.5 \quad , \text{ where } L = 0.1\ \text{cm}$$

$$\alpha_{max} = 5\ \text{cm}^{-1}$$

$$\alpha_{max} = \frac{h\nu}{c\Delta\nu}(N_2 - N_1)B_{12}$$

$$B_{12} = \frac{c^3}{8\pi h\nu^3}A_{21}$$

$$\alpha_{max} = \frac{\cancel{h\nu}}{\cancel{c}\Delta\nu}(N_2 - N_1)\frac{\overset{c^2}{\cancel{c^3}}}{8\pi\cancel{h\nu^3}_{\nu^2}}A_{21}$$

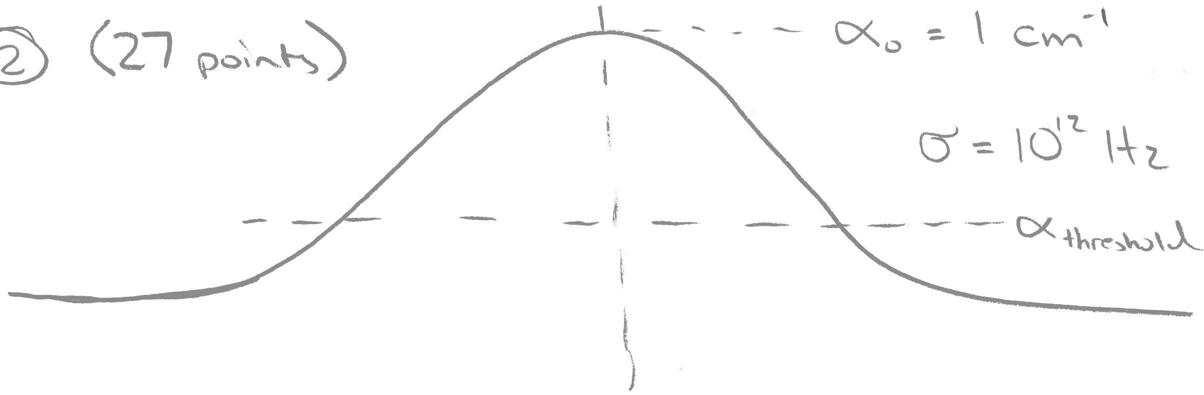
$$\frac{8\pi\Delta\nu\,\alpha_{max}}{\underset{\lambda^2}{c^2/\nu^2}} = (N_2 - N_1)A_{21}$$

$$\frac{8\pi(10^{12}\ \text{Hz})(5\ \text{cm}^{-1})}{(8\times10^{-5}\ \text{cm})^2} = (N_2 - N_1)A_{21}$$

$$\boxed{1.97\times10^{22}\ \frac{1}{\text{cm}^3\text{-s}} = (N_2 - N_1)A_{21}}$$

This is the transition rate density of the excited state to the ground state.

② (27 points)



(a.) $FSR = \dfrac{c}{2nL} = \dfrac{\cancel{3}\times 10^8\ m/s}{\cancel{2(1.5)}(10^{-2} m)} = 10^{10}\ Hz$

$\alpha_{threshold} 2L = \delta$

$\alpha_{threshold} = \dfrac{0.2707}{2(10^0 cm)} = 0.13535\ cm^{-1}$

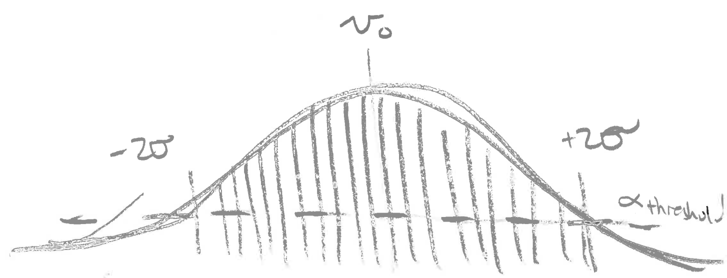
$\alpha_{threshold} = \alpha_0 e^{-\frac{1}{2}\left(\frac{\nu-\nu_0}{\sigma}\right)^2}$

$-2\ln\left(\dfrac{\alpha_{threshold}}{\alpha_0}\right) = \left(\dfrac{\nu-\nu_0}{\sigma}\right)^2$

$\pm\sigma\sqrt{-2\ln\left(\dfrac{\alpha_{threshold}}{\alpha_0}\right)} = \nu-\nu_0$

$\pm 10^{12} Hz\sqrt{-2\ln\left(\dfrac{0.13535\ cm^{-1}}{1\ cm^{-1}}\right)} = \nu-\nu_0$

$\pm\ 2\times 10^{12}\ Hz = \pm 2\sigma = \nu - \nu_0$



So between -2σ and $+2\sigma$ on our gain curve, we can achieve lasing. That freq. span, 4×10^{12} Hz, supports $\dfrac{4\times10^{12}\ Hz}{10^{10}\ Hz}$ = $\boxed{400 \text{ modes.}}$

(b.) Linewidth is proportional to $\sqrt{T}$. We need a reduction of 400 in σ to achieve a single mode so: $\dfrac{1}{400} = \dfrac{\sigma_T}{\sigma_{300K}} = \sqrt{\dfrac{T}{300K}} \Rightarrow \dfrac{300K}{400^2} = T$

$\boxed{T \approx 1.875\ mK.}$

(c.) No, this is a terrible way to change oscillating mode #. 1.88 mK is very difficult to obtain, for example. A better way would be to increase the cavity length.

③ (30 points)

(a.)

$$\hat{\sigma}_{++} = \langle +|\sigma|+\rangle = 1$$

$$\hat{\sigma}_{+-} = \langle +|\sigma|-\rangle = 0$$

$$\hat{\sigma}_{-+} = \langle -|\sigma|+\rangle = 0$$

$$\hat{\sigma}_{--} = \langle -|\sigma|-\rangle = -1$$

$$\boxed{\hat{\sigma} = |+\rangle\langle +| - |-\rangle\langle -| \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(b.) $\hat{\rho}_\theta = |\theta\rangle\langle\theta| = \frac{1}{\sqrt{2}}\left[|+\rangle + e^{i\theta}|-\rangle\right]\left[\langle +| + \langle -|e^{-i\theta}\right]\frac{1}{\sqrt{2}}$

$$\boxed{\hat{\rho}_\theta = \frac{1}{2}\left[|+\rangle\langle +| + |-\rangle\langle -| + e^{i\theta}|-\rangle\langle +| + e^{-i\theta}|+\rangle\langle -|\right]}$$

(c.) $\langle\sigma\rangle = \mathrm{Tr}\{\hat{\rho}_\theta\hat{\sigma}\}$

$$= \mathrm{Tr}\left\{\frac{1}{2}\left[|+\rangle\langle +| + |-\rangle\langle -| + e^{i\theta}|-\rangle\langle +| + e^{-i\theta}|+\rangle\langle -|\right]\left[|+\rangle\langle +| - |-\rangle\langle -|\right]\right\}$$

$$= \mathrm{Tr}\left\{\frac{1}{2}\left[|+\rangle\langle +| - |-\rangle\langle -| + e^{i\theta}|-\rangle\langle +| - e^{-i\theta}|+\rangle\langle -|\right]\right\}$$

$\langle\sigma\rangle = \frac{1}{2} - \frac{1}{2} = \boxed{0}$ (since $|n\rangle\langle n|$ are the only elements on the diagonal)

(d.) To determine this, we need to compute $\hat{\rho}_\theta^2$ and find the trace. This can most easily be done by using matrix multiplication:

$$\hat{\rho}_\theta = \frac{1}{2}\begin{bmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{bmatrix} \Rightarrow \hat{\rho}_\theta^2 = \frac{1}{4}\begin{bmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{bmatrix}\begin{bmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{bmatrix}$$

$$\hat{\rho}_\theta^2 = \frac{1}{4}\begin{bmatrix} 2 & 2e^{-i\theta} \\ 2e^{i\theta} & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{bmatrix}$$

$$\mathrm{Tr}\{\hat{\rho}_\theta^2\} = 1 \Rightarrow \boxed{\text{pure state}}$$

$$\mathrm{Tr}\{\hat{\rho}_\theta\} = 1$$

(e.) $$\hat{\rho}_1 = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Tr}\{\hat{\rho}_1\} = 1$$

$$\hat{\rho}_1^2 = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Tr}\{\hat{\rho}_1^2\} = \frac{1}{2} \rightarrow \boxed{\text{mixed state}}$$

4 (21 points)

(a.) No, b/c the Zeeman splitting $\gg$ thermal energy, all the spins are polarized in the ground state.

(b.) No, since this a two-level system, we cannot make this into a laser.

(c.) Once the field is removed, the system will return to equilibrium @ a characteristic time T_1.