PHYS 4310, Mid-Term 2, Monday November 23rd

1. [16 points] A positively charged trion (one electron and two positive electrons, also known as holes) are in a solid with a dielectric constant $\epsilon = 10\epsilon_0$. In this particular solid, the mass of an electron is $0.1m_e$, while the mass of a hole is m_e.
(a) What is the (estimated) Bohr radius of the trion in nanometers?
(b) What is the binding energy of the trion in terms of the hydrogen binding energy, E_B^H ($= 13.6$ eV)?

(a.) $$a_0^T = \frac{4\pi\epsilon\hbar^2}{\mu Z e^2} \Rightarrow a_0^T = a_0 \underbrace{\left[\frac{\epsilon}{\epsilon_0}\right]}_{10}\underbrace{\left[\frac{1}{Z}\right]}_{\frac{1}{2}}\underbrace{\left[\frac{m_e}{\mu}\right]}_{10} = 50a_0$$

where $$\mu = \frac{\overbrace{(0.1m_e)}^{\text{effective } e^- \text{ mass}}\overbrace{2m_e}^{\text{holes}}}{0.1m_e + 2m_e} = \frac{0.2}{2.1}m_e \simeq \frac{m_e}{10}$$

$$a_0^T = 50a_0 = 50(0.529\,\text{Å}) = 5(0.529\,\text{nm}) \cong \boxed{2.6\text{ nm}}$$

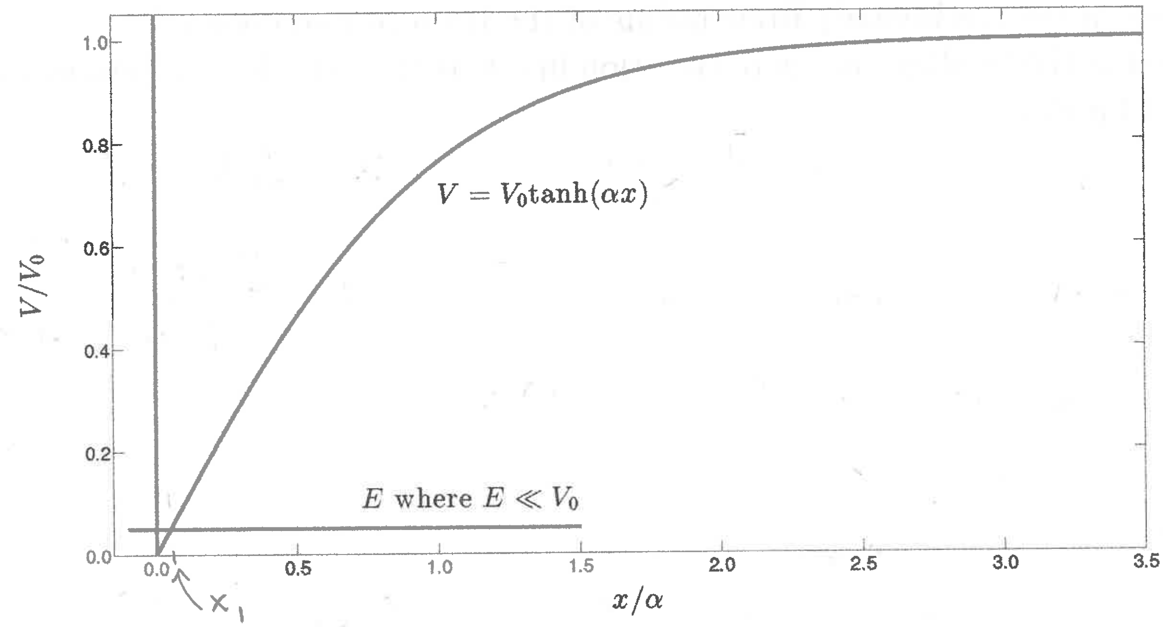
(b.) $$E_1^H = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -13.6\text{ eV}$$

$$E_1^T = \left[\underbrace{\left(\frac{\mu}{m_e}\right)}_{\frac{1}{10}}\underbrace{\left(\frac{Z^2}{1^2}\right)}_{2^2}\underbrace{\left(\frac{\epsilon_0}{\epsilon}\right)^2}_{\left(\frac{1}{10}\right)^2}\right]E_1^H = \frac{E_1^H}{250}$$

$$= \frac{4}{1000} = \frac{1}{250}$$

$\therefore$ The binding energy of the trion is $\frac{E_B^H}{250}$ or 250 times less than hydrogen.

2. [20 points] Consider a potential, $V(x)$, such that: $V(x) = \begin{cases} \infty & x \leq 0 \\ V_0 \tanh(\alpha x) & x > 0, \end{cases}$ where $V_0 > 0$ (see figure). Using the WKB approximation, find the bound energy states, E_n. Assume that $E \ll V_0$ and make approximations where appropriate.



$\int_0^{x_1} p(x)\,dx = (n-\frac{1}{4})\pi\hbar$ b/c there is one vertical wall

$p(x) = \sqrt{2m(E-V(x))}$, where $E = V_0 \tanh(\alpha x_1)$

But since $E \ll V_0$ so $\tanh(\alpha x_1) \simeq \alpha x_1$ $\left[\frac{e^{\alpha x}-e^{-\alpha x}}{e^{\alpha x}+e^{-\alpha x}}\right]_{x_1} \simeq \frac{\alpha x_1 + \alpha x_1}{2} = \alpha x_1$

$\int_0^{x_1} \sqrt{2m}\left(V_0\alpha x_1 - V_0\alpha x\right)^{1/2} dx = \sqrt{2mV_0\alpha}\int_0^{x_1}(x_1-x)^{1/2}dx = \sqrt{2mV_0\alpha}\left[-\frac{2}{3}(x_1-x)^{3/2}\right]_0^{x_1}$

$\sqrt{2mV_0\alpha}\,\frac{2}{3}x_1^{3/2} = (n-1/4)\pi\hbar$, w/ $E \simeq V_0\alpha x_1 \rightarrow \frac{E}{V_0\alpha} = x_1$

$\sqrt{2mV_0\alpha}\left(\frac{E_n}{V_0\alpha}\right)^{3/2} = \frac{3\pi\hbar}{2}(n-1/4) \Rightarrow \frac{\sqrt{2m}}{V_0\alpha}E_n^{3/2} = \frac{3\pi\hbar}{2}(n-1/4)$

$$\boxed{E_n = \left[\frac{3\pi\hbar V_0\alpha}{\sqrt{8m}}(n-1/4)\right]^{2/3}}$$ which is only valid while E_n remains in the linear regime of $V(x)$, i.e., $E_n \ll V_0$.

3. [10 points] Consider a particle in the infinite three-dimensional well. An operator, $\hat{A}$, is defined such that it only depends on $\hat{p}^2$. What is difference between the expectation value, $\langle A \rangle$, at t and $t + t_0$?

$$\frac{d\langle A\rangle}{dt} = \frac{-1}{i\hbar}\langle [\hat{H}, \hat{A}] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

Since $\hat{A}$ depends only on $\hat{p}^2$ and $V=0$,

$[\hat{H}, \hat{A}] = 0$. By that same logic, $\hat{A}$'s

dependence on $\hat{p}^2$ precludes an explicit dependence

on time, so $\frac{\partial A}{\partial t} = 0$.

$\therefore \frac{d\langle A\rangle}{dt} = 0$ so $\langle A\rangle$ @ t and $t+t_0$

are the same.

4. [25 points] The three-dimensional Fourier transform of $\psi(r,\theta,\phi)$ to $\Phi(\vec{p})$ for spherically symmetric ψ is:

$$\Phi(\vec{p}) = \frac{4\pi}{\sqrt{(2\pi\hbar)^3}} \int_0^\infty \psi(r) \left[\frac{\sin(pr/\hbar)}{pr/\hbar}\right] r^2 dr. \tag{1}$$

Find the $\Phi(\vec{p})$ for $\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$ (the ground state of a hydrogenic atom with an atomic number Z, where $a = \frac{a_0}{Z}$). Simplify your solution so that it is in the form $C_0\left[\frac{1}{1+C_1^2}\right]^2$, where C_0 is a constant and C_1 is a function of p.

$$\Phi(p) = \frac{\cancel{4\pi}^{\,2}}{\cancel{2\pi\hbar}\sqrt{2\pi\hbar}} \frac{\cancel{\hbar}}{p} \frac{1}{a\sqrt{\pi a}} \int_0^\infty r e^{-r/a} \frac{1}{2i}\left[e^{ipr/\hbar} - e^{-ipr/\hbar}\right] dr$$

$$= \frac{1}{2i\pi p a}\sqrt{\frac{2}{\hbar a}}\left[\int_0^\infty r e^{-\frac{1}{a}\left(1-\frac{ipa}{\hbar}\right)r} dr - \int_0^\infty r e^{-\frac{1}{a}\left(1+\frac{ipa}{\hbar}\right)r} dr\right]$$

$$\alpha = \frac{1}{a}\left(1-\frac{ipa}{\hbar}\right) \qquad \beta = \frac{1}{a}\left(1+\frac{ipa}{\hbar}\right)$$

$$\begin{array}{ccc} r & \oplus & e^{-\alpha r} \\ 1 & \ominus & -\frac{1}{\alpha}e^{-\alpha r} \\ 0 & & \frac{1}{\alpha^2}e^{-\alpha r} \end{array} \Bigg\} \text{ and same for } \beta \text{ term}$$

$$= \frac{1}{2i\pi p a}\sqrt{\frac{2}{a\hbar}}\left[-\frac{r}{\alpha}e^{-\alpha r} - \frac{1}{\alpha^2}e^{-\alpha r} + \frac{r}{\beta}e^{-\beta r} + \frac{1}{\beta^2}e^{-\beta r}\right]_0^\infty$$

$$= \frac{1}{2i\pi p a}\sqrt{\frac{2}{a\hbar}}\left[\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right] = \frac{1}{2i\pi p a}\sqrt{\frac{2}{a\hbar}}\left[\frac{\beta^2-\alpha^2}{(\alpha\beta)^2}\right]$$

$$\beta^2 - \alpha^2 = \frac{1}{a^2}\left[1 + \frac{2ipa}{\hbar} - \frac{p^2a^2}{\hbar^2} - 1 + \frac{2ipa}{\hbar} + \frac{p^2a^2}{\hbar^2}\right] = \frac{1}{a^2}\frac{4ipa}{\hbar}$$

$$(\alpha\beta)^2 = \frac{1}{a^4}\left[\left(1-\frac{ipa}{\hbar}\right)\left(1+\frac{ipa}{\hbar}\right)\right]^2 = \frac{1}{a^4}\left(1+\frac{p^2a^2}{\hbar^2}\right)^2$$

$$= \frac{1}{\cancel{2i\pi pa}}\sqrt{\frac{2}{a\hbar}}\left[\frac{\frac{1}{a^2}\frac{\cancel{4}^{\,2} ipa}{\hbar}}{\frac{1}{a^4}\left(1+\frac{p^2a^2}{\hbar^2}\right)^2}\right] = \frac{1}{\pi}\sqrt{\frac{2}{a\hbar}}\,\frac{2a^2}{\hbar}\left[\frac{1}{1+\left(\frac{pa}{\hbar}\right)^2}\right]^2$$

$$\boxed{\Phi(\bar{p}) = \frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2}\left[\frac{1}{1+\left(\frac{pa}{\hbar}\right)^2}\right]^2}$$ for the ψ_{100} state in hydrogen