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4.19

(a) $[L_z, x] = [\underbrace{xp_y - yp_x}, x] = [\underset{AB,\ C}{xp_y, x}] - [yp_x, x]$

$L_z = xp_y - yp_x$

$$[L_z, x] = \underbrace{\left(x[p_y, x] + \cancel{[x, x]}p_y\right)} - \left(y[p_x, x] + \cancel{[y, x]}p_x\right)$$

$[r_i, r_j] = 0$

$[xp_y, x] \rightarrow [AB, C] = A[B,C] + [A,C]B$

$[r_i, p_j] = -\underbrace{[p_i, r_j]} = i\hbar\delta_{ij}$

$$[L_z, x] = x\cancel{[p_y, x]} - y[p_x, x] = \boxed{i\hbar y}$$

$$[L_z, y] = [xp_y, y] - [yp_x, y]$$

$$= \left(x\underbrace{[p_y, y]}_{-i\hbar} + \cancel{[x, y]}p_y\right) - \left(y\cancel{[p_x, y]} + \cancel{[y, y]}p_x\right)$$

$[L_z, y] = -i\hbar x$

$$[L_z, z] = [xp_y, z] - [yp_x, z]$$

$$= \left(x\cancel{[p_y, z]} + \cancel{[x, z]}p_y\right) - \left(y\cancel{[p_x, z]} + \cancel{[y, z]}p_x\right)$$

$[L_z, z] = 0$

$$[L_z, p_x] = \left(x\cancel{[p_y, p_x]} + \cancel{[x, p_x]}p_y\right) - \left(y\cancel{[p_x, p_x]} + \cancel{[y, p_x]}p_x\right)$$

$[p_i, p_j] = 0$ $\qquad i\hbar\delta_{ij}$

$[L_z, p_x] = i\hbar p_y$

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$$[L_z, p_y] = \left(x\underbrace{[p_y, p_y]}_{0} + \underbrace{[x, p_y]}_{0}p_y\right) - \left(y\underbrace{[p_x, p_y]}_{0} + \underbrace{[y, p_y]}_{i\hbar}p_x\right)$$

$$[L_z, p_y] = -i\hbar p_x$$

$$[L_z, p_z] = \left(x\underbrace{[p_y, p_z]}_{0} + \underbrace{[x, p_z]}_{0}p_y\right) - \left(y\underbrace{[p_x, p_z]}_{0} + \underbrace{[y, p_z]}_{0}p_x\right)$$

$$[L_z, p_z] = 0$$

(b.) $$[L_z, L_x] = [L_z, (yp_z - zp_y)]$$

$$= \underbrace{[L_z, yp_z]}_{①} - \underbrace{[L_z, zp_y]}_{②}$$

$$[AB, C] = A[B, C] + [A, C]B$$

① $$[L_z, yp_z] = -[yp_z, L_z] = -\left(y\underbrace{[p_z, L_z]}_{0} + \underbrace{[y, L_z]}_{-(-i\hbar x)}p_z\right)$$

$$= -i\hbar x p_z$$

② $$-[L_z, zp_y] = [zp_y, L_z] = \left(z\underbrace{[p_y, L_z]}_{-(-i\hbar p_x)} + \underbrace{[z, L_z]}_{0}p_y\right)$$

$$= i\hbar z p_x$$

$$[L_z, L_x] = -i\hbar(\underbrace{\overset{①}{xp_z} - \overset{②}{zp_x}}_{-L_y}) = i\hbar L_y$$

(c.) $$[L_z, r^2] = [L_z, x^2 + y^2 + z^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2]$$

$$= -\left\{[x^2, L_z] + [y^2, L_z] + [z^2, L_z]\right\}$$

$$= -\left\{\left(x[x, L_z] + [x, L_z]x\right) + \left(y[y, L_z] + [y, L_z]y\right) + \left(z[z, L_z] + [z, L_z]z\right)\right\}$$

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$[L_z, r^2] = -\{(x[x, L_z] + [x, L_z]x) + (y[y, L_z] + [y, L_z]y) +$
$(z[z, L_z] + [z, L_z]z)\}$

$[x, L_z] = -[L_z, x] = -i\hbar y$

$[y, L_z] = -[L_z, y] = i\hbar x$

$[z, L_z] = -[L_z, z] = 0$

$[L_z, r^2] = -\{(x(-i\hbar y) + (-i\hbar y)x) + (y(i\hbar x) + (i\hbar x)y)\}$

$= -i\hbar(-xy - yx + yx + xy) = 0$

$[L_z, p^2] = -\{(p_x[p_x, L_z] + [p_x, L_z]p_x) + (p_y[p_y, L_z] + [p_y, L_z]p_y) +$
$(p_z[p_z, L_z] + [p_z, L_z]p_z)\}$

where I've used $p^2 = p_x^2 + p_y^2 + p_z^2$ and just substituted p_i in my previous result for $r^2 = x^2 + y^2 + z^2$

$[L_z, p^2] = -\{(p_x(-i\hbar p_y) + (-i\hbar p_y)p_x) + (p_y(i\hbar p_x) + (i\hbar p_x)p_y)\}$

$= -i\hbar(-p_x p_y - p_y p_x + p_y p_x + p_x p_y)$

$[L_z, p^2] = 0$

(d.) $H = \frac{p^2}{2m} + V(r) = \frac{p^2}{2m} + V(\sqrt{r^2})$

In part (c.), we found that L_z commutes w/ p^2 and r^2, so $[L_z, H] = 0$. However, the same logic in (c.) that held for L_z also holds for L_x and L_y, so $[L_x, H] = [L_y, H]$.

$\therefore [L_x + L_y + L_z, H] = [L, H] = 0$ QED

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4.20

(a.)

Let's look @ the L_x component of L

$$\frac{d}{dt}\langle L_x\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{L}_x]\rangle + \cancel{\left\langle\frac{\partial L_x}{\partial t}\right\rangle}^{0}$$

$$[\hat{H},\hat{L}_x] = \frac{1}{2}[\hat{p}^2,\hat{L}_x] + [V,L_x]$$

From 4.19(c.), we have $[\hat{p}^2,\hat{L}_i]=0$, so the 1st term is 0

$$[V,L_x] = [V, yp_z - zp_y] = y[V,p_z] - z[V,p_y]$$

From 3.13(c.), $[f(x),p] = i\hbar\frac{\partial f}{\partial x}$ for any func. of x

$\therefore$ $[V,p_z] = i\hbar\frac{\partial V}{\partial z}$ and $[V,p_y] = i\hbar\frac{\partial V}{\partial y}$

$$[V,L_x] = yi\hbar\frac{\partial V}{\partial z} - zi\hbar\frac{\partial V}{\partial y} = i\hbar[\bar{r}\times\nabla V]_x$$

$$\frac{d\langle L_x\rangle}{dt} = \frac{i}{\hbar}i\hbar\langle(\bar{r}\times\nabla V)_x\rangle = -\langle\bar{r}\times\nabla V\rangle_x$$

Since the same procedure can be used ~~of~~ for the y and z components, we have

$$\frac{d\langle\bar{L}\rangle}{dt} = -\langle\bar{r}\times\nabla V\rangle = \langle\bar{r}\times(-\nabla V)\rangle = \langle\bar{N}\rangle \quad \text{QED}$$

(b.)

For a spherically symmetric potential,

$V(\bar{r}) = V(r)\hat{r}$, so $\nabla V = \frac{\partial V}{\partial r}\hat{r}$

But $\bar{r}\times\frac{\partial V}{\partial r}\hat{r} = 0$ since $\hat{r}\times\hat{r}=0$, so $\frac{d\langle\bar{L}\rangle}{dt} = 0$.

4.21

(a.) $$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_+L_-f = -\hbar^2\left[e^{+i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right]f\right)\right]$$

$$= -\hbar^2\left\{e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]\left[e^{-i\phi}\frac{\partial f}{\partial\theta} - ie^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right]\right\}$$

$$= -\hbar^2\left\{e^{i\phi}\left[e^{-i\phi}\frac{\partial^2 f}{\partial\theta^2} - ie^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\partial\theta}\cot\theta - \frac{\partial f}{\partial\phi}\csc^2\theta\right) + i\cot\theta\left(-ie^{-i\phi}\frac{\partial f}{\partial\theta} + e^{-i\phi}\frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right.\right.$$

$$\left.\left. + \cot^2\theta\left(-ie^{-i\phi}\frac{\partial f}{\partial\phi} + e^{-i\phi}\frac{\partial^2 f}{\partial\phi^2}\right)\right]\right\}$$

$$= -\hbar^2\left\{\frac{\partial^2 f}{\partial\theta^2} \cancel{- i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}} \cancel{+ i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}} + i\left(\csc^2\theta\frac{\partial f}{\partial\phi} - \cot^2\theta\frac{\partial f}{\partial\phi}\right) + \right.$$

$$\left. \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right\}$$

$$\underbrace{\frac{1}{\sin^2\theta}}_{\csc^2\theta} - \underbrace{\frac{\cos^2\theta}{\sin^2\theta}}_{\cot^2\theta} = 1$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + i\frac{\partial}{\partial\phi} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right\}f$$ dropping the test function

$$L_+L_- = -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right\}$$ QED

(b.) $$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$L^2 = L_+L_- + L_z^2 - \hbar L_z = -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right\} +$$

$$(-\hbar^2)\frac{\partial^2}{\partial\phi^2} + i\hbar^2\frac{\partial}{\partial\phi}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \cancel{+ i\frac{\partial}{\partial\phi}} + \frac{\partial^2}{\partial\phi^2} \cancel{- i\frac{\partial}{\partial\phi}}\right\}$$

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$\cos^2\theta + \sin^2\theta = 1$

$$L^2 = -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \left(\frac{\overbrace{\cos^2\theta + \sin^2\theta}^{1}}{\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2}\right\}$$

$$= -\hbar^2\left\{\underbrace{\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

this term can be written as: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}$

$$\therefore L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad \text{QED}$$

4.22

(a.) $L_+ Y_\ell^\ell = 0$ since this is the highest rung of the ladder $(m \le |\ell|)$.

(b.)

$L_+ Y_\ell^\ell = 0$, where $L_+ = \hbar e^{+i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$

$L_z Y_\ell^\ell = \hbar\ell Y_\ell^\ell$ (same as $L_z Y_\ell^m = m\hbar Y_\ell^m$, just w/ $\ell = m$)

But $L_z = -i\hbar\frac{\partial}{\partial\phi}$ so: $-i\hbar\frac{\partial Y_\ell^\ell}{\partial\phi} = \hbar\ell Y_\ell^\ell$

acts like a constant in this DE

$$\frac{\partial Y_\ell^\ell}{\partial\phi} = i\ell Y_\ell^\ell \rightarrow Y_\ell^\ell = g(\theta)e^{i\ell\phi}$$

$$L_+ Y_\ell^\ell = 0 \Rightarrow \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)g(\theta)e^{i\ell\phi} = 0$$

$$\frac{dg}{d\theta}e^{i\ell\phi} - \ell\cot\theta\, g(\theta)e^{i\ell\phi} = 0$$

$$\frac{dg}{d\theta} = \ell\cot\theta\, g$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\int\frac{dg}{g} = \int \ell\cot\theta\, d\theta = \ell\int\cos\theta\frac{d\theta}{\sin\theta} = \ell\int\frac{du}{u}$$

$$\ln g = \ell\ln u + C = \ell\ln(\sin\theta) + C$$

$$\ln g = \ln(\sin^\ell\theta) + C$$

$$g(\theta) = A\sin^{\ell}\theta$$

$$\therefore Y_\ell^\ell = g(\theta)e^{i\ell\phi} = \boxed{A(\sin\theta e^{i\phi})^\ell}$$

(c.)

$$1 = \int_0^{2\pi}\int_0^{\pi} |Y_\ell^\ell|^2 \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} |A|^2 \sin^{2\ell}\theta \sin\theta\, e^{-i\phi} e^{i\phi}\, d\phi$$

$$= 2\pi|A|^2 \int_0^{\pi} \sin^{2\ell}\theta \sin\theta\, d\theta = 2\pi A^2 \int_0^{\pi} \sin^{(2\ell+1)}\theta\, d\theta$$

$$= 2\pi|A|^2\, 2\left[\frac{2\cdot4\cdot6\cdot8\cdot \ldots 2\ell}{1\cdot3\cdot5\cdot7\cdot\ldots(2\ell+1)}\right] =$$

In order to make the numerator a squared quantity, we can multiply the top and bottom by $2\cdot4\cdot6\cdot8 \ldots 2\ell$ to get:

$$1 = 4\pi|A|^2 \frac{(2\cdot4\cdot6 \ldots 2\ell)^2}{(2\ell+1)!} =$$

$$2\cdot4\cdot6\cdot8 \ldots 2\ell = (2\cdot1)\cdot(2\cdot2)\cdot(2\cdot3)\cdot(2\cdot4)\ldots 2\cdot\ell$$

$$= 2^\ell \ell!$$

$$1 = 4\pi A^2 \frac{(2^\ell \ell!)^2}{(2\ell+1)!} \longrightarrow A = \frac{1}{2\,2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{\pi}} = \boxed{\frac{1}{2^{\ell+1}\ell!}\sqrt{\frac{(2\ell+1)!}{\pi}}}$$

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4

$$\bar{J} \equiv \frac{i\hbar}{2\mu}\left(\psi \nabla \psi^* - \psi^* \nabla \psi\right)$$

(a.) $\psi_{211} = \frac{1}{\sqrt{24}} a_0^{-3/2} \frac{r}{a_0} \exp(-r/2a_0)\left(-\left[\frac{3}{8\pi}\right]^{1/2} \sin\theta e^{i\phi}\right)$

$$= -\frac{1}{8\sqrt{\pi}}\sqrt{\frac{1}{a_0^5}}\, r\sin\theta e^{-r/2a_0} e^{i\phi}$$

$$\nabla\psi = \frac{-1}{8a_0^2\sqrt{\pi a_0}}\left[\sin\theta e^{i\phi}\left(1-\frac{r}{2a}\right)e^{-r/2a}\hat{r} + \frac{r e^{-r/2a_0}}{r} e^{i\phi}\cos\theta\hat{\theta} + \frac{i}{r\sin\theta} r e^{-r/2a_0} \sin\theta e^{i\phi}\hat{\phi}\right]$$

$$\nabla\psi = \frac{-1}{8a_0^2\sqrt{\pi a_0}}\left[\sin\theta\left(1-\frac{r}{2a_0}\right)\hat{r} + \cos\theta\hat{\theta} + i\hat{\phi}\right]e^{-\frac{r}{2a_0}}e^{i\phi}$$

Similarly:

$$\nabla\psi^* = \frac{-1}{8a_0^2\sqrt{\pi a_0}}\left[\sin\theta\left(1-\frac{r}{2a_0}\right)\hat{r} + \cos\theta\hat{\theta} - i\hat{\phi}\right]e^{-\frac{r}{2a_0}}e^{-i\phi}$$

① $\psi\nabla\psi^* = \frac{r\sin\theta}{64a_0^5\pi}\left[\sin\theta\left(1-\frac{r}{2a_0}\right)\hat{r} + \cos\theta\hat{\theta} - i\hat{\phi}\right]e^{-\frac{r}{a_0}}$

② $\psi^*\nabla\psi = \frac{r\sin\theta}{64a_0^5\pi}\left[\sin\theta\left(1-\frac{r}{2a_0}\right)\hat{r} + \cos\theta\hat{\theta} + i\hat{\phi}\right]e^{-\frac{r}{a_0}}$

Subtracting gives:

$$① - ② = \frac{-2r\sin\theta i}{64a_0^5\pi} e^{-\frac{r}{a_0}}\hat{\phi}$$

$$\bar{J} = \frac{i\hbar}{2\mu}\left(\frac{-2ir\sin\theta}{64\pi a_0^5} e^{-r/a_0}\hat{\phi}\right) = \boxed{\frac{\hbar}{64\pi\mu a_0^5} r\sin\theta e^{-r/a_0}\hat{\phi}}$$

(b.) $\bar{L} = \mu\int \bar{r}\times\bar{J}\, d^3\bar{r}$

$$\bar{r}\times\bar{J} = \frac{\hbar}{64\pi\mu a_0^5} r^2\sin\theta e^{-r/a_0}\underbrace{(\hat{r}\times\hat{\phi})}_{-\hat{\theta}}$$

1 3 = -2

Since we want L_z, we must find the z^{th} component of $\bar{r}\times\bar{J}$. $\bar{r}\times\bar{J}$ is in the $-\hat{\theta}$ direction, so the z^{th} component of $\bar{r}\times\bar{J}$ is just $-\hat{z}\cdot\hat{\theta}$, which is:

$$-(\hat{z}\cdot\hat{\theta}) = -\underbrace{(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}_{\hat{z}}\cdot\hat{\theta} = -(-\sin\theta) = \sin\theta$$

$$L_z = \cancel{\mu}\int_V \frac{\hbar}{64\pi\cancel{\mu} a_0^5}\, r^2\sin^2\theta\, e^{-r/a_0}\, dV = \frac{2\pi\hbar}{\underset{32}{\cancel{64}}\pi a_0^5}\int_0^{\pi}\int_0^{\infty} r^4\sin^3\theta\, e^{-r/a_0}\, dr\, d\theta$$

$$= \frac{\hbar}{32a_0^5}\underbrace{\int_0^{\infty} r^4 e^{-r/a_0}\, dr}_{②}\ \underbrace{\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta}_{①}$$

①
$$\int_0^{\pi}\sin\theta\, d\theta = -\cos\theta\Big|_0^{\pi} = 2$$

$$-\int_0^{\pi}\sin\theta\cos^2\theta\, d\theta \qquad \begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array}$$

$$-\int_{-1}^{1} u^2\, du = -\frac{u^3}{3}\Big|_{-1}^{1} = -\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = -\frac{2}{3}$$

$$2 - \frac{2}{3} = \frac{4}{3}$$

②
$$\int_0^{\infty} r^4 e^{-\frac{r}{a_0}}\, dr =$$

r^4		e^{-r/a_0}
$4r^3$	$+$	$-a_0 e^{-r/a_0}$
$12r^2$	$-$	$a_0^2 e^{-r/a_0}$
$24r$	$+$	$-a_0^3 e^{-r/a_0}$
24	$-$	$a_0^4 e^{-r/a_0}$
0	$+$	$-a_0^5 e^{-r/a_0}$

~~Sin~~ All e^{-r/a_0} terms go to 0 faster than r^n goes to ∞ when $r\to\infty$. Similarly, all r^n $(n\geq 1)$ terms go to 0 when $r\to 0$.

$$\therefore \int_0^{\infty} r^4 e^{-r/a_0}\, dr = 24a_0^5$$

$$L_z = \frac{\hbar}{\cancel{32a_0^5}} \underbrace{\overset{8}{\cancel{24}}\,a_0^5}_{2} \underbrace{\frac{4}{3}}_{1} = \hbar$$

This is exactly the result we expect, since

$$\hat{L}_z |\ell\ m\rangle = m\hbar |\ell\ m\rangle$$

$$\hat{L}_z |1\ \ 1\rangle = \hbar |1\ \ 1\rangle \quad \text{for } \psi_{211}$$