3.13

(a.) $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$

$$= \hat{A}\underbrace{(\hat{B}\hat{C} - \hat{C}\hat{B})}_{[\hat{B},\hat{C}]} + \cancel{\hat{A}\hat{C}\hat{B}} - \underbrace{(\hat{C}\hat{A} - \hat{A}\hat{C})}_{[\hat{C},\hat{A}]}\hat{B} - \cancel{\hat{A}\hat{C}\hat{B}}$$

$$= \hat{A}[\hat{B},\hat{C}] - [\hat{C},\hat{A}]\hat{B}$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} \quad \text{QED}$$

(b.) $[x^n, p] = [x^n, -i\hbar\frac{\partial}{\partial x}] = -i\hbar\left(x^n\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x^n\right)$

$$-i\hbar\left(x^n\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x^n\right)\underbrace{f(x)}_{\uparrow} = -i\hbar\left[\cancel{x^n\frac{\partial f}{\partial x}} - nx^{n-1}f(x) - \cancel{x^n\frac{\partial f}{\partial x}}\right]$$

test function

$$= i\hbar n x^{n-1} f(x)$$

Removing the test function gives us our result:

$$[x^n, p] = i\hbar n x^{n-1} \quad \text{QED}$$

(c.)

$$[f(x), p] = [f(x), -i\hbar\frac{\partial}{\partial x}]$$

$$[f(x), -i\hbar\frac{\partial}{\partial x}]g(x) = -i\hbar\left(f(x)\frac{\partial g}{\partial x} - \frac{\partial}{\partial x}(fg)\right)$$

$$= -i\hbar\left(\cancel{fg'} - f'g - \cancel{fg'}\right)$$

$$= i\hbar\frac{\partial f}{\partial x}g$$ → Removing our test function gives:

$$[f(x), p] = i\hbar\frac{\partial f}{\partial x} \quad \text{QED}$$

(d.) $[\hat{A},\hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

$$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \cancel{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \cancel{\hat{B}\hat{A}\hat{C}}$$

$$[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}] \quad \text{QED}$$

3.28 $$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iE_nt/\hbar} \quad \text{where } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Phi_n(p,t) = \sqrt{\frac{1}{2\pi\hbar}}\int \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iE_nt/\hbar}e^{-ipx/\hbar}\,dx$$

$$= \sqrt{\frac{1}{\pi\hbar a}}\,e^{-iE_nt/\hbar}\int_0^a e^{-ipx/\hbar}\left(e^{\frac{in\pi}{a}x} - e^{-\frac{in\pi}{a}x}\right)\frac{dx}{2i}$$

$$= \sqrt{\frac{1}{\pi\hbar a}}\,e^{-iE_nt/\hbar}\,\frac{1}{2i}\int_0^a \left[e^{i\left(\frac{n\pi}{a}-\frac{p}{\hbar}\right)x} - e^{-i\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)x}\right]dx$$

$$= \sqrt{\frac{1}{\pi\hbar a}}\,e^{-iE_nt/\hbar}\,\frac{1}{2i}\left[\frac{1}{i\left(\frac{n\pi}{a}-\frac{p}{\hbar}\right)}\left[e^{i\left(\frac{n\pi}{a}-\frac{p}{\hbar}\right)a}-1\right] + \frac{1}{i\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)}\left[e^{-i\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)a}-1\right]\right]$$

$$= \sqrt{\frac{1}{\pi\hbar a}}\,e^{-iE_nt/\hbar}\left(-\frac{1}{2}\right)\left\{\frac{a}{n\pi-\frac{pa}{\hbar}}\left[e^{i\left(n\pi-\frac{pa}{\hbar}\right)}-1\right] + \frac{a}{n\pi+\frac{pa}{\hbar}}\left[e^{-i\left(n\pi+\frac{pa}{\hbar}\right)}-1\right]\right\}$$

$$e^{in\pi} = \cos(n\pi) + i\cancelto{0}{\sin(n\pi)} = (-1)^n, \text{ so:}$$

$$\Phi_n(p,t) = \sqrt{\frac{1}{\pi\hbar a}}\,e^{-iE_nt/\hbar}\left(\frac{-a}{2}\right)\left\{\frac{(-1)^n e^{-i\frac{pa}{\hbar}}-1}{n\pi-\frac{pa}{\hbar}} + \frac{(-1)^n e^{-i\frac{pa}{\hbar}}-1}{n\pi+\frac{pa}{\hbar}}\right\}$$

$$= \sqrt{\frac{a}{\pi\hbar}}\,e^{-iE_nt/\hbar}\left(\frac{-1}{2}\right)\left\{\left[(-1)^n e^{-i\frac{pa}{\hbar}}-1\right]\left[\frac{\cancel{2}n\pi}{(n\pi)^2-\left(\frac{pa}{\hbar}\right)^2}\right]\right\}$$

$$\boxed{\Phi_n(p,t) = \sqrt{\frac{a\pi}{\hbar}}\,e^{-iE_nt/\hbar}\left[1-(-1)^n e^{-ipa/\hbar}\right]\frac{n}{(n\pi)^2-\left(\frac{pa}{\hbar}\right)^2}}$$

Sketches of $|\Phi_1|^2$ and $|\Phi_2|^2$

First, let's write $\Phi_1(p,t)$ and $\Phi_2(p,t)$ in more reasonable forms:

$$\Phi_1(p,t) = \sqrt{\frac{a\pi}{\hbar}}\, e^{-iE_1 t/\hbar}\left[1 + e^{-ipa/\hbar}\right]\frac{1}{\pi^2 - (pa/\hbar)^2}$$

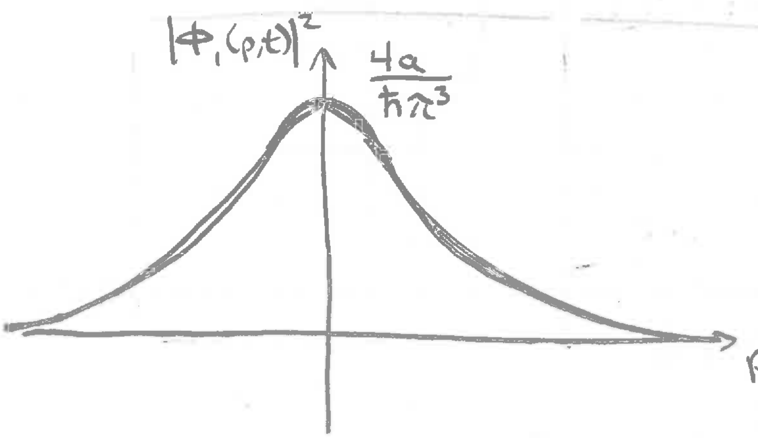
$$= \sqrt{\frac{a\pi}{\hbar}}\, e^{-iE_1 t/\hbar}\, e^{-i\frac{pa}{2\hbar}}\underbrace{\left[e^{+i\frac{pa}{2\hbar}} + e^{-i\frac{pa}{2\hbar}}\right]}_{2\cos\left(\frac{pa}{2\hbar}\right)}\frac{1}{\pi^2 - (pa/\hbar)^2}$$

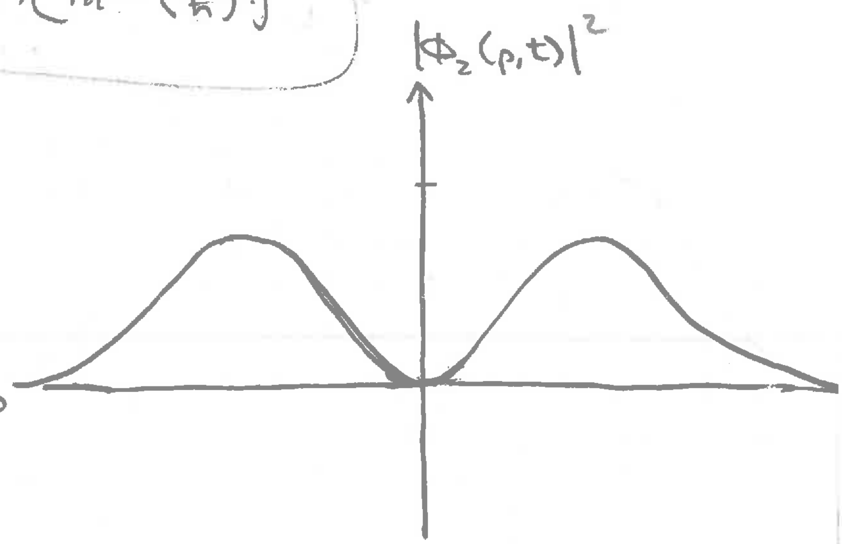
$$= 2\sqrt{\frac{a\pi}{\hbar}}\, e^{-iE_1 t/\hbar}\, e^{-i\frac{pa}{2\hbar}}\cos\left(\frac{pa}{2\hbar}\right)\frac{1}{\pi^2 - (pa/\hbar)^2}$$

So: $$\boxed{|\Phi_1(p,t)|^2 = 4\,\frac{a\pi}{\hbar}\cos^2\left(\frac{pa}{2\hbar}\right)\left[\frac{1}{\pi^2 - (pa/\hbar)^2}\right]^2}$$

$$\Phi_2(p,t) = \sqrt{\frac{a\pi}{\hbar}}\, e^{-iE_2 t/\hbar}\, e^{-i\frac{pa}{2\hbar}}\underbrace{\left[e^{+i\frac{pa}{2\hbar}} - e^{-i\frac{pa}{2\hbar}}\right]}_{2i\sin\left(\frac{pa}{2\hbar}\right)}\frac{2}{(2\pi)^2 - \left(\frac{pa}{\hbar}\right)^2}$$

$$\boxed{|\Phi_2(p,t)|^2 = 16\,\frac{a\pi}{\hbar}\sin^2\left(\frac{pa}{2\hbar}\right)\left[\frac{1}{4\pi^2 - \left(\frac{pa}{\hbar}\right)^2}\right]^2}$$





3.25

① $|1\rangle = 1 \qquad \langle 1|1\rangle = \int_{-1}^{1} dx = 2 \rightarrow \boxed{|1'\rangle = \sqrt{\tfrac{1}{2}} \ \left(=\sqrt{\langle 1'|1'\rangle}\right)}$

② $|2\rangle = x \qquad \langle 1'|2\rangle = \sqrt{\tfrac{1}{2}}\int_{-1}^{1} x\,dx = \sqrt{\tfrac{1}{2}}\left(\frac{x^2}{2}\Big|_{-1}^{1}\right) = 0$

$$\langle 2'|2'\rangle^{1/2} = \langle 2|2\rangle^{1/2} = \left[\int_{-1}^{1} x^2 dx\right]^{1/2} = \left[\frac{x^3}{3}\Big|_{-1}^{1}\right]^{1/2} = \sqrt{\frac{2}{3}} \Rightarrow \boxed{|2'\rangle = \sqrt{\frac{3}{2}}\,x}$$

③

$$|3'\rangle = |3\rangle - \langle 1'|3\rangle|1'\rangle - \langle 2'|3\rangle|2'\rangle$$

where $|3\rangle = x^2$

$$\langle 1'|3\rangle = \sqrt{\frac{1}{2}}\int_{-1}^{1} x^2 dx = \sqrt{\frac{1}{2}}\left[\frac{x^3}{3}\Big|_{-1}^{1}\right] = \cancel{\sqrt{\tfrac{2}{3}}}\ \frac{\sqrt{2}}{3}$$

$$\langle 2'|3\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^3 dx = 0$$

$$|3'\rangle = x^2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}} = x^2 - 1/3$$

$$\langle 3'|3'\rangle^{1/2} = \left[\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx\right]^{1/2} = \left[\int_{-1}^{1}\left[x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right]dx\right]^{1/2} = \left[\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{x}{9}\Big|_{-1}^{1}\right]^{1/2}$$

$$= \left[\frac{2}{5} - \frac{4}{9} + \frac{2}{9}\right]^{1/2} = \left[\frac{18-20+10}{45}\right]^{1/2} = \sqrt{\frac{8}{45}}$$

$$\boxed{|3'\rangle = \sqrt{\frac{45}{8}}\left[x^2 - \frac{1}{3}\right] = \sqrt{\frac{5}{8}}\left[3x^2-1\right] = \sqrt{\frac{5}{2}}\left[\frac{3x^2-1}{2}\right]}$$

④ $|4\rangle = x^3$

$$|4'\rangle = |4\rangle - \langle 1'|4\rangle|1'\rangle - \langle 2'|4\rangle|2'\rangle - \langle 3'|4\rangle|3'\rangle$$

$$\langle 1'|4\rangle = \sqrt{\frac{1}{2}}\int_{-1}^{1} x^3 dx = \sqrt{\frac{1}{2}}\ \frac{x^4}{4}\Big|_{-1}^{1} = 0,$$

$$\langle 2'|4\rangle = \sqrt{\tfrac{3}{2}}\int_{-1}^{1} x^4\,dx = \frac{2}{5}\sqrt{\frac{3}{2}}$$

$$\langle 3'|4\rangle = \frac{1}{2}\sqrt{\frac{5}{2}}\int_{-1}^{1}(3x^5 - x^3)\,dx = \frac{1}{2}\sqrt{\frac{5}{2}}\left[\frac{3x^6}{6} - \frac{x^4}{4}\Bigg|_{-1}^{1}\right] = 0$$

$$|4'\rangle = x^3 - \left(\frac{2}{5}\sqrt{\frac{3}{2}}\right)\sqrt{\frac{3}{2}}\,x = x^3 - \frac{3}{5}x$$

$$\sqrt{\langle 4'|4'\rangle} = \left[\int_{-1}^{1}\left(x^3 - \frac{3}{5}x\right)^2 dx\right]^{1/2} = \left[\int_{-1}^{1}\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)dx\right]^{1/2}$$

$$= \left[\frac{x^7}{7} - \frac{6}{25}x^5 + \frac{9}{75}x^3\Bigg|_{-1}^{1}\right]^{1/2} = \left[\frac{2}{7} - \frac{12}{25} + \frac{\cancel{18}\ 6}{\cancel{75}\ 25}\right]^{1/2} = \left[\frac{2}{7} - \frac{6}{25}\right]^{1/2} = \sqrt{\frac{8}{7\cdot 25}}$$

$$\boxed{|4'\rangle = \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right) = \sqrt{\frac{7}{2}}\left(\frac{5x^3 - 3x}{2}\right)}$$

4. $f(x) = x^{\nu}$

Condition for Hilbert space: $\langle f|f\rangle$ over 0 to 1

$$\int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1} x^{2\nu+1}\Big|_0^1 = \frac{1}{2\nu+1}\left(1-0^{2\nu+1}\right)$$

$0^{\alpha} = 0$ if $\alpha = 2\nu+1>0$, so $\nu > -\frac{1}{2}$

if $\alpha = 2\nu+1<0$, $\nu < -1/2$, $0^{\alpha}\to\infty$

But what about $\nu = -1/2$?

$$\int_0^1 x^{2(-\frac{1}{2})}\,dx = \int_0^1 x^{-1}\,dx = \ln(x)\Big|_0^1 = \ln 1 - \ln 0 = \infty$$

So $\nu = -1/2$ is not in the Hilbert Space.

$\therefore$ ~~which~~ $f(x) = x^{\nu}$ is in Hilbert space for $\nu = -1/2$ over $x[0,1]$.

(b) For $\nu = -1/2$, $f(x)$ is not in this Hilbert space

$xf(x) = x^{1/2}$, so yes, this is in the Hilbert space.

$\frac{d}{dx}(x^{\nu})$ for $\nu = -1/2 \rightarrow -\frac{1}{2}x^{-3/2}$. Since $\nu = -3/2$, this is not in the Hilbert space

5. $|\psi_I\rangle = e^{iH_0t/\hbar}|\psi_s\rangle$

$Q_I(t) = e^{iH_0t/\hbar}\hat{Q}_s e^{-iH_0t/\hbar}$

$$\frac{d}{dt}|\psi_I\rangle = iH_0/\hbar\, \underbrace{e^{iH_0t/\hbar}|\psi_s\rangle}_{|\psi_I\rangle} + e^{iH_0t/\hbar}\frac{\partial}{\partial t}|\psi_s\rangle$$

But $i\hbar\frac{\partial}{\partial t}|\psi_s\rangle = H|\psi_s\rangle$ (SE), so:

$$\frac{d}{dt}|\psi_I\rangle = iH_0/\hbar\,|\psi_I\rangle + e^{iH_0t/\hbar}\frac{1}{i\hbar}H|\psi_s\rangle$$

$$\frac{d}{dt}|\psi_I\rangle = \frac{-H_0}{i\hbar}|\psi_I\rangle + \frac{1}{i\hbar}(\underbrace{H_0 + H_I'}_{H(t)})|\psi_I\rangle$$

$$= \frac{1}{i\hbar} H_I' |\psi_I\rangle$$

$$\boxed{i\hbar \frac{d}{dt}|\psi_I\rangle = H_I'|\psi_I\rangle}$$

QED

$$\frac{d}{dt} Q_I(t) = \frac{d}{dt}\Big[\underbrace{e^{iH_0t/\hbar}\,\hat{Q}_s\,e^{-iH_0t/\hbar}}_{\hat{Q}_I}\Big]$$

$$= iH_0/\hbar\,\hat{Q}_I + e^{iH_0t/\hbar}\frac{\partial \hat{Q}_s}{\partial t}e^{-iH_0t/\hbar} - \hat{Q}_I\frac{iH_0}{\hbar}$$

$$= -\frac{1}{i\hbar}H_0\hat{Q}_I + \frac{1}{i\hbar}\hat{Q}_I H_0 + \frac{\partial Q_I}{\partial t}$$

$$\boxed{\frac{d}{dt}Q_I = \frac{1}{i\hbar}[\hat{Q}_I, H_0] + \frac{\partial Q_I}{\partial t}}$$

QED