4.18 (15 pts)

$$L_\pm f_\ell^m = (A_\ell^m) f_\ell^{m\pm1}$$

$$\langle f_\ell^m | L_\mp L_\pm f_\ell^m \rangle = \langle f_\ell^m | L^2 - L_z^2 \mp \hbar L_z | f_\ell^m \rangle$$

$$= \langle f_\ell^m | L^2 | f_\ell^m \rangle - \langle f_\ell^m | L_z^2 | f_\ell^m \rangle \mp \hbar \langle f_\ell^m | L_z | f_\ell^m \rangle$$

$$= \langle f_\ell^m | \hbar^2 [\ell(\ell+1)] - \hbar^2 m^2 \mp \hbar^2 m | f_\ell^m \rangle$$

$$= \hbar^2 [\ell(\ell+1) - m(m\pm1)] \cancel{\langle f_\ell^m | f_\ell^m \rangle}^{\nearrow 1}$$

$$\langle L_\pm f_\ell^m | L_\pm f_\ell^m \rangle = \hbar^2 [\ell(\ell+1) - m(m\pm1)]$$

$$\langle A_\ell^m f_\ell^{m\pm1} | A_\ell^m f_\ell^{m\pm1} \rangle =$$

$$|A_\ell^m|^2 \cancel{\langle f_\ell^{m\pm1} | f_\ell^{m\pm1} \rangle}^{\nearrow 1} =$$

$$|A_\ell^m|^2 = \hbar^2 [\ell(\ell+1) - m(m\pm1)]$$

$$\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)} \qquad \text{QED}}$$

(20 pts)

2.
$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u = Eu$$

$$V(r) = \frac{-Ze^2}{r}$$

$$u(r) = (Ar + Br^2)e^{-br}$$

$$\frac{du}{dr} = (A + 2Br)e^{-br} - b(Ar + Br^2)e^{-br}$$

$$\frac{d^2u}{dr^2} = 2Be^{-br} + b^2(Ar + Br^2)e^{-br} - (Ab + 2bBr)e^{-br}$$
$$- b(A + 2Br)e^{-br}$$
$$= e^{-br}\left[(2B - 2bA) + (b^2A - 4bB)r + b^2Br^2\right]$$

$$\frac{-\hbar^2}{2m}\left[\cancel{e^{-br}}\left([2B - 2bA] + [b^2A - 4bB]r + b^2Br^2\right)\right] - Ze^2(A + Br)\cancel{e^{-br}} =$$

$$E(Ar + Br^2)\cancel{e^{-br}}$$

Equating like powers of r:

$\underline{r^0}$
$$-\frac{\hbar^2}{2m}[2B - 2bA] - Ze^2A = 0$$

$\underline{r^1}$
$$-\frac{\hbar^2}{2m}[b^2A - 4bB] - Ze^2B = EA$$

$\underline{r^2}$
$$\frac{-\hbar^2}{2m}b^2B = EB \Rightarrow b^2 = \frac{-2m}{\hbar^2}E$$

$\underline{r^1}$

$$\frac{-\hbar^2}{2m}\left[b^2A - 4bB\right] - Ze^2B = EA$$

substituting in: $b^2 = \frac{-2m}{\hbar^2}E$ gives:

$$\cancel{E}A + \frac{4bB\hbar^2}{2m} - Ze^2B = \cancel{E}A$$

$$4b\cancel{B}\hbar^2 = 2mZe^2\cancel{B}$$

$$b = \frac{mZe^2}{2\hbar^2} = \frac{Ze^2m}{2\hbar^2}$$

$$\boxed{b = \frac{Ze^2m}{2\hbar^2}}$$

$\underline{r^0}$

$$\frac{-\hbar^2}{2m}\left[2B - 2bA\right] - Ze^2A = 0$$

4b

$$2B - 2bA = \frac{-Ze^2 2m}{\hbar^2}A$$

$$2B = -2bA$$

$$B = -bA \rightarrow \boxed{b = -B/A}$$

(b.)

$$a_0 = \frac{\hbar^2}{me^2}$$

$$B/A = -b = \frac{-Ze^2m}{2\hbar^2} = \boxed{\frac{-Z}{2}\frac{1}{a_0}}$$

$b^2 = \left(\frac{Ze^2m}{2\hbar^2}\right)^2$ (from our excmination of r^1 terms)

$b^2 = \frac{-2m}{\hbar^2}E$ (" " " " " " " r^2 terms)

$$\underbrace{-\frac{2m}{\hbar^2}E}_{b^2} = \overbrace{\left(\frac{Ze^2m}{2\hbar^2}\right)^2}^{b^2}$$

$$-\cancel{\frac{2m}{\hbar^2}}E = -\frac{Z^2e^4m^{\cancel{2}}}{4\hbar^{\cancel{4}\,2}}$$

$$\boxed{E = -\frac{Z^2}{4}\boxed{\frac{e^4m}{2\hbar^2}}}$$

Ry = ground state of hydrogen

(c.) $\boxed{b = -B/A = \frac{Ze^2m}{2\hbar^2} = \frac{Z}{2a_0}}$

3. (25 pts)

$$\psi_{T,100}(r,\theta,\phi) = \sqrt{\frac{1}{4\pi}}\sqrt{\frac{\cancel{4}}{a_0^3}}\, e^{-r/a_0} = \sqrt{\frac{1}{\pi a_0^3}}\, e^{-r/a_0}$$

$$\psi_{He^{+},100}(r,\theta,\phi) = \sqrt{\frac{1}{4\pi}}\left(\frac{4}{a^3}\right)^{1/2} e^{-r/a} = \sqrt{\frac{8}{\pi a_0^3}}\, e^{-2r/a_0}$$

$a = \frac{a_0}{Z}$ where $Z=2$

$$P_{T\to He} = |\langle \psi_{He,100}|\psi_{T,100}\rangle|^2 = \left|\int_V \left[\frac{1}{\pi a_0^3}\sqrt{2}\, e^{-3r/a_0}\right] dV\right|^2$$

$$= \left|\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\sqrt{8}}{\pi a_0^3} e^{-3r/a_0} r^2 \sin\theta \, dr\, d\theta\, d\phi\right|^2$$

$$= \left|\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} \frac{\sqrt{8}}{\pi a_0^3} e^{-3r/a_0} r^2\, dr\right|^2$$

$$= \left|\frac{2\sqrt{8}}{a_0^3}\left[-\cos\theta\Big|_0^{\pi}\right]\int_0^{\infty} r^2 e^{-3r/a_0}\, dr\right|^2 = \left|\frac{4\sqrt{8}}{a_0^3}\int_0^{\infty} r^2 e^{-3r/a_0}\, dr\right|^2$$

$$\begin{array}{lcl}
r^2 & & e^{-3r/a_0} \\
2r & (+) & -\frac{a_0}{3} e^{-3r/a_0} \\
2 & (-) & \frac{a_0^2}{9} e^{-3r/a_0} \\
0 & (+) & -\frac{a_0^3}{27} e^{-3r/a_0}
\end{array}$$

$$= \left\{\frac{4\sqrt{8}}{a_0^3}\left[-\frac{a_0 r^2}{3} e^{-3r/a_0} - \frac{2a_0^2 r}{9} e^{-3r/a_0} - \frac{2a_0^3}{27} e^{-3r/a_0}\right]_0^{\infty}\right\}^2 = \left[\frac{4\sqrt{8}}{a_0^3}\,\frac{2a_0^3}{27}\right]^2$$

$$= \left(\frac{8\sqrt{8}}{27}\right)^2 = \boxed{0.70}$$

(b.) $\psi_{He^{\oplus}}(r,\theta,\phi)$ in $n=1, \ell=1, m=0$:

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \text{ , so } \psi_{He^{\oplus}} = \sqrt{\frac{3}{4\pi}}\cos\theta \sqrt{\frac{4}{a^3}}\, e^{-r/a}$$

$$a = \frac{a_0}{Z} = \frac{a_0}{2} \Rightarrow \psi_{He^{\oplus}} = \sqrt{\frac{24}{\pi a_0^3}}\cos\theta\, e^{-2r/a_0}$$

$$P = |\langle \psi_{He^{\oplus}} | \psi_T \rangle|^2 = \left[\frac{\sqrt{24}}{\pi a_0^3}\int_0^{2\pi} d\phi \int_0^{\pi} \underbrace{\sin\theta\cos\theta\, d\theta}_{\frac{1}{2}\sin 2\theta} \int_0^{\infty} r^2 e^{-3r/a_0}\, dr\right]^2$$

$$= \frac{2\sqrt{24}}{a_0^3}\underbrace{\int_0^{\pi}\frac{1}{2}\sin 2\theta\, d\theta}_{\frac{-1}{4}\cos 2\theta\big|_0^{\pi} = 0}\int_0^{\infty} r^2 e^{-3r/a_0}\, dr$$

$$\boxed{P = 0}$$

4. (20 pts)

$$\lambda(\lambda+1) = \ell(\ell+1) + \frac{2mA}{\hbar^2}$$

$$V(r) = \frac{A}{r^2} + \frac{B}{r} \qquad A, B > 0$$

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left(\frac{B}{r} + \frac{\hbar^2\ell(\ell+1) + 2mA}{2mr^2}\right)u = Eu$$

$$E = \frac{-\hbar^2k^2}{2m} \longrightarrow k = \frac{\sqrt{-2mE}}{\hbar}$$

$$\frac{E}{k^2}\frac{d^2u}{dr^2} = u\left(E - \frac{B}{r} - \frac{\hbar^2\ell(\ell+1) + 2mA}{2mr^2}\right)$$

$$\frac{1}{k^2}\frac{d^2u}{dr^2} = u\left(1 + \frac{2mB}{\hbar^2k^2r} + \frac{\ell(\ell+1) + 2mA/\hbar^2}{k^2r^2}\right)$$

As w/ our previous route: $\rho = kr$

$$\rho_0 = \frac{-2mB}{\hbar^2 k}$$

Using the hint given: $\lambda(\lambda+1) = \frac{2mA}{\hbar^2} + \ell(\ell+1)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\lambda(\lambda+1)}{\rho^2}\right]u$$

Following the procedure outlined in the book gives:

$$E_k = \frac{-\hbar^2k^2}{2m} \quad \text{where} \quad k^2 = \frac{4m^2B^2}{\rho_0^2\hbar^4}$$

$$E = -\frac{\hbar^2}{2m}\left(\frac{4m^2B^2}{\rho_0^2\hbar^4}\right) = \frac{-2mB^2}{\rho_0^2\hbar^2}$$

Taking the numerator of our series soln and capping the highest value of j @ j_{max} such that $2(j_{max}+\lambda+1)-\rho_0=0$:

$$\rho_0^2 = \left[2(j_{max}+\lambda+1)\right]^2 \rightarrow 4(j_{max}+\lambda+1)^2$$

$$E_{n,l} = \frac{-2mB^2}{\hbar^2\rho_0^2} = \frac{-2mB^2}{4\hbar^2(j+\lambda+1)^2} = \frac{-mB^2}{2\hbar^2(n+\lambda+1)^2}$$

The only difference in $E_{n,l}$ from the hydrogen case is $\frac{A}{r^2}$, which impacts our values of λ.

$$j_{max} = n-(\lambda+1) \rightarrow \lambda = n-1 \text{ for } j_{max}=0$$

so $\lambda = 0, 1, \ldots, n-1$ just as w/ l for $A=0$.

But if λ is an integer, then l is not, which means m (required to be an integer) is limited to only a single value. Thus, the degeneracy is broken by adding a second term to the potential.