$$\psi_{210} = \frac{1}{4a^{5/2}}\sqrt{\frac{1}{2\pi}}\, r e^{-r/2a} \cos\theta\, e^{-i\omega_1 t}$$

$$\psi_{211} = \frac{-1}{8a^{5/2}}\sqrt{\frac{1}{\pi}}\, r e^{-r/2a} \sin\theta\, e^{-i(\omega_2 t - \varphi)}$$

$$\psi = \sqrt{\frac{1}{2}}\left(\psi_{210} + \psi_{211}\right)$$

$$\psi = \underbrace{\boxed{\frac{1}{8a^{5/2}}\sqrt{\frac{1}{\pi}}}}_{A}\ \underbrace{\boxed{r e^{-r/2a}}}_{R(r)}\left[\cos\theta\, e^{-i\omega_1 t} - \sqrt{\frac{1}{2}}\sin\theta\, e^{-i(\omega_2 t - \varphi)}\right]$$

$$\psi^* = A R(r)\left[\cos\theta\, e^{i\omega_1 t} - \sqrt{\frac{1}{2}}\sin\theta\, e^{i(\omega_2 t - \varphi)}\right]$$

$$\bar{J}_e(\bar{r},t) = \frac{-ie\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$$

$$\bar{\mu}_m(t) = \int_V \bar{r}\times\bar{J}_e(\bar{r},t)\, dV$$

$$\bar{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$$

Let's first calculate our operator $\bar{r}\times\bar{\nabla}$

$$\bar{r}\times\bar{\nabla} = r\hat{r}\times\bar{\nabla} = \frac{\partial}{\partial\theta}\hat{\varphi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\hat{\theta}, \text{ where } \hat{r}\times\hat{r} = 0$$

$$\hat{r}\times\hat{\theta} = \hat{\varphi}$$

$$\hat{r}\times\hat{\varphi} = -\hat{\theta}$$

$$\boxed{\bar{r}\times\bar{\nabla} = \frac{-1}{\sin\theta}\frac{\partial}{\partial\varphi}\hat{\theta} + \frac{\partial}{\partial\theta}\hat{\varphi}}$$

This says that $R(r)$ can be treated as a constant until we calculate $\int_V \bar{r}\times\bar{J}_e(\bar{r},t)\, dV$

Calculation of $\bar{r}\times\bar{\bar{J}}_e(\bar{r},t)$:

First, let's break up the problem a little:

$$\bar{r}\times\bar{\bar{J}}_e(\bar{r},t) = \frac{-ie\hbar}{2m}\left[\overset{(1)}{\psi(\bar{r}\times\nabla\psi^*)} - \overset{(2)}{\psi^*(\bar{r}\times\nabla\psi)}\right]$$

We can do this b/c ψ and ψ^* both act like constants in this situation

$$\bar{r}\times\bar{\nabla}\psi^* = AR(r)\left[\frac{-1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(Y_1^0 e^{i\omega_1 t} + Y_1^1 e^{i\omega_2 t}\right)\hat{\theta} + \frac{\partial}{\partial\theta}\left(Y_1^0 e^{i\omega_1 t} + Y_1^1 e^{i\omega_2 t}\right)\hat{\varphi}\right]$$

$$= AR(r)\left[-i\sqrt{\frac{1}{2}}\,\overset{1}{\cancel{\frac{\sin\theta}{\sin\theta}}}\,e^{i(\omega_2 t-\varphi)}\hat{\theta} + \left(-\sin\theta\, e^{i\omega_1 t} - \sqrt{\frac{1}{2}}\cos\theta\, e^{i(\omega_2 t-\varphi)}\right)\hat{\varphi}\right]$$

$$\bar{r}\times\bar{\nabla}\psi = AR(r)\left[i\sqrt{\frac{1}{2}}\,\underset{1}{\cancel{\frac{\sin\theta}{\sin\theta}}}\,e^{-i(\omega_2 t-\varphi)}\hat{\theta} + \left(-\sin\theta\, e^{-i\omega_1 t} - \sqrt{\frac{1}{2}}\cos\theta\, e^{-i(\omega_2 t-\varphi)}\right)\hat{\varphi}\right]$$

---

(1) $$\psi(\bar{r}\times\nabla\psi^*) = A^2R^2(r)\left[\cos\theta\, e^{-i\omega_1 t} - \sqrt{\frac{1}{2}}\sin\theta\, e^{-i(\omega_2 t-\varphi)}\right]\left[-i\sqrt{\frac{1}{2}}\,e^{i(\omega_2 t-\varphi)}\hat{\theta} + \left(-\sin\theta\, e^{i\omega_1 t} - \sqrt{\frac{1}{2}}\cos\theta\, e^{i(\omega_2 t-\varphi)}\right)\hat{\varphi}\right]$$

$$= A^2R^2(r)\left\{\left[\frac{-i}{\sqrt{2}}\cos\theta\, e^{i(\omega_{21}t-\varphi)} + \frac{i}{2}\sin\theta\right]\hat{\theta} + \left[-\cos\theta\sin\theta - \frac{\cos^2\theta}{\sqrt{2}}e^{i(\omega_{21}t-\varphi)} + \frac{\sin^2\theta}{\sqrt{2}}e^{-i(\omega_{21}t-\varphi)} + \frac{\sin\theta\cos\theta}{2}\right]\hat{\varphi}\right\}$$

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(2) $$\psi^*(\bar{r}\times\nabla\psi) = A^2R^2(r)\left[\cos\theta\, e^{i\omega_1 t} - \sqrt{\frac{1}{2}}\sin\theta\, e^{i(\omega_2 t-\varphi)}\right]\left[\frac{i}{\sqrt{2}}e^{-i(\omega_2 t-\varphi)}\hat{\theta} + \left(-\sin\theta\, e^{-i\omega_1 t} - \sqrt{\frac{1}{2}}\cos\theta\, e^{-i(\omega_2 t-\varphi)}\right)\hat{\varphi}\right]$$

$$= A^2R^2(r)\left\{\left[\frac{i\cos\theta}{\sqrt{2}}e^{-i(\omega_{21}t-\varphi)} - \frac{i}{2}\sin\theta\right]\hat{\theta} + \left[-\cos\theta\sin\theta - \frac{\cos^2\theta}{\sqrt{2}}e^{-i(\omega_{21}t-\varphi)} + \frac{\sin^2\theta}{\sqrt{2}}e^{i(\omega_{21}t-\varphi)} + \frac{\cos\theta\sin\theta}{2}\right]\hat{\varphi}\right\}$$

③

$$\bar{r} \times \bar{J}_e(\bar{r},t) = \frac{-ie\hbar}{2m} A^2 R^2(r) \left\{ \left[\frac{-i\cos\theta}{\sqrt{2}} \left(e^{i(\omega_{21}t-\varphi)} + e^{-i(\omega_{21}t-\varphi)} \right) + i\sin\theta \right] \hat{\theta} \right.$$

$$\left[\cancel{-\cos\theta\sin\theta} + \cancel{\cos\theta\sin\theta} - \frac{\cos^2\theta}{\sqrt{2}} \left(e^{i(\omega_{21}t-\varphi)} - e^{-i(\omega_{21}t-\varphi)} \right) + \frac{\sin^2\theta}{\sqrt{2}} \left(e^{-i(\omega_{21}t-\varphi)} - e^{i(\omega_{21}t-\varphi)} \right) \right.$$

$$\left. \left. + \cancel{\frac{\sin\theta\cos\theta}{2}} - \cancel{\frac{\sin\theta\cos\theta}{2}} \right] \hat{\varphi} \right\}$$

$$\boxed{\bar{r} \times \bar{J}_e(\bar{r},t) = \frac{e\hbar}{2m} A^2 R^2(r) \left\{ \left[-\sqrt{2}\cos\theta\cos(\omega_{21}t-\varphi) + \sin\theta \right] \hat{\theta} + \left[-\sqrt{2}\sin(\omega_{21}t-\varphi) \right] \hat{\varphi} \right\}}$$

$$\bar{\mu}_m(t) = \int_V \bar{r} \times \bar{J}_e(\bar{r},t)\, dV = \frac{e\hbar}{2m} A^2 \underbrace{\int_{r=0}^{\infty} r^2 R^2(r)}_{①} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \left\{ \underbrace{\left[-\sqrt{2}\cos\theta\cos(\omega_{21}t-\varphi) + \sin\theta \right]}_{②} \hat{\theta} + \underbrace{\left[-\sqrt{2}\sin(\omega_{21}t-\varphi) \right]}_{③} \hat{\varphi} \right\}$$

$$① \int_0^\infty r^2 R^2(r)\, dr = \int_0^\infty r^4 e^{-r/a}\, dr \longrightarrow$$

$$\begin{array}{ccc} r^4 & & e^{-r/a} \\ 4r^3 & + & -ae^{-r/a} \\ 12r^2 & - & a^2 e^{-r/a} \\ 24r & + & -a^3 e^{-r/a} \\ 24 & - & a^4 e^{-r/a} \\ 0 & + & -a^5 e^{-r/a} \end{array}$$

Since we are evaluating our terms from $r=0$ to ∞, any (anti-derivative) term w/ r in it is 0 @ $r=0$. Similarly, $e^{-r/a} \to 0$ @ $r=\infty$. Therefore, the only term that survives is $-24a^5 e^{-r/a}$ evaluated @ ∞ (0) and 0 $(24a^5)$.

$$\therefore \int_0^\infty r^4 e^{-r/a}\, dr = 24a^5 = ①$$

2

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left[-\sqrt{2}\cos\theta\cos(\omega_{21}t-\varphi)+\sin\theta\right]\hat{\theta}\,\sin\theta\,d\theta\,d\varphi$$

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{x}+\cos\theta\sin\varphi\,\hat{y}-\sin\theta\,\hat{z}$$

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left[-\sqrt{2}\cos\theta\sin\theta\cos(\omega_{21}t-\varphi)+\sin^2\theta\right]\left[\cos\theta\cos\varphi\,\hat{x}+\cos\theta\sin\varphi\,\hat{y}-\sin\theta\,\hat{z}\right]d\theta\,d\varphi$$

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\Big\{\left[-\sqrt{2}\cos^2\theta\sin\theta\cos\varphi\cos(\omega_{21}t-\varphi)+\underbrace{\cancel{\sin^2\theta\cos\theta\cos\varphi}}_{0\ (\varphi\ \text{integral})}\right]\hat{x}\ +$$

$$\left[-\sqrt{2}\cos^2\theta\sin\theta\sin\varphi\cos(\omega_{21}t-\varphi)+\underbrace{\cancel{\sin^2\theta\cos\theta\sin\varphi}}_{0\ (\varphi\ \text{integral})}\right]\hat{y}\ +$$

$$\left[\underbrace{\cancel{\sqrt{2}\cos\theta\sin^2\theta\cos(\omega_{21}t-\varphi)}}_{0\ (\varphi\ \text{integral})}-\sin^3\theta\right]\hat{z}\,d\theta\,d\varphi$$

Let's find the θ integrals first:

$$\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = \int_{-1}^{1}x^2\,dx\ ,\ \text{where}\ \ x=\cos\theta,\ \ dx=-\sin\theta\,d\theta$$

$$= \frac{x^3}{3}\Big|_{-1}^{1} = \boxed{\frac{2}{3}}$$

$$\int_0^{\pi}\sin^3\theta\,d\theta = \int_0^{\pi}\sin\theta\,d\theta - \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = -\cos\theta\Big|_0^{\pi}-\frac{2}{3}$$

$$= 2-\frac{2}{3} = \boxed{\frac{4}{3}}$$

Now our integral looks like:

$$-\frac{2\sqrt{2}}{3}\hat{x}\int_{\varphi=0}^{2\pi}\cos\varphi\cos(\omega_{21}t-\varphi)\,d\varphi + \frac{-2\sqrt{2}}{3}\hat{y}\int_{\varphi=0}^{2\pi}\sin\varphi\cos(\omega_{21}t-\varphi)\,d\varphi -$$

$$\frac{4}{3}\hat{z}\int_{\varphi=0}^{2\pi}d\varphi$$

$\cos(\omega_{21}t-\varphi)$ can be broken down as: $\cos\omega_{21}t\cos\varphi+\sin\omega_{21}t\sin\varphi$

$$-\frac{2\sqrt{2}}{3}\hat{x}\int_0^{2\pi}\left(\cos^2\varphi\cos\omega_{21}t + \cancel{\cos\varphi\sin\varphi\sin\omega_{21}t}^{\,0}\right)d\varphi - \frac{2\sqrt{2}}{3}\hat{y}\int_0^{2\pi}\left(\cancel{\sin\varphi\cos\varphi\cos\omega_{21}t}^{\,0}\right.$$

$$\left.+\sin^2\varphi\sin\omega_{21}t\right)d\varphi - \frac{8\pi}{3}\hat{z}$$

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta = \pi$$

$\therefore$ 2 is equal to $\boxed{-\frac{2\pi\sqrt{2}}{3}\cos\omega_{21}t\,\hat{x} - \frac{2\pi\sqrt{2}}{3}\sin\omega_{21}t\,\hat{y} - \frac{8\pi}{3}\hat{z}}$

3

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left[-\sqrt{2}\sin\theta\sin(\omega_{21}t-\varphi)\right]\hat{\varphi}\,d\theta\,d\varphi$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x}+\cos\varphi\,\hat{y}$$

$$\int_{\theta=0}^{\pi}\sin\theta\,d\theta = -\cos\theta\Big|_0^{\pi} = 2$$

$$-2\sqrt{2}\int_{\varphi=0}^{2\pi} \sin(\omega_{21}t-\varphi)\left[-\sin\varphi\,\hat{x}+\cos\varphi\,\hat{y}\right]d\varphi$$

$$\sin(\omega_{21}t-\varphi)=\sin\omega_{21}t\cos\varphi-\cos\omega_{21}t\sin\varphi$$

$$-2\sqrt{2}\int_{\varphi=0}^{2\pi}\Big[-\sin\omega_{21}t\cancelto{0}{\sin\varphi\cos\varphi}\,\hat{x}+\cos\omega_{21}t\sin^2\varphi\,\hat{x}+\sin\omega_{21}t\cos^2\varphi\,\hat{y}$$

$$-\cos\omega_{21}t\cancelto{0}{\cos\varphi\sin\varphi}\,\hat{y}\Big]d\varphi$$

$$\boxed{3 = -2\pi\sqrt{2}\left[\cos\omega_{21}t\,\hat{x}+\sin\omega_{21}t\,\hat{y}\right]}$$

Putting this altogether:

$$\bar{\mu}_m(t)=\frac{e\hbar}{2m}\underbrace{\frac{1}{\underset{8}{64}\pi a^5}\overset{3}{24a^5}}_{A^2}\Big\{\underbrace{-\frac{2\pi\sqrt{2}}{3}\cos\omega_{21}t\,\hat{x}-\frac{2\pi\sqrt{2}}{3}\sin\omega_{21}t\,\hat{y}}_{1}$$

$$\underbrace{-\frac{8\pi}{3}\hat{z}}_{2}\underbrace{-2\pi\sqrt{2}\cos\omega_{21}t\,\hat{x}-2\pi\sqrt{2}\sin\omega_{21}t\,\hat{y}}_{3}\Big\}$$

$$=\frac{-3e\hbar}{16m}\Big\{\underbrace{\left(\frac{2\sqrt{2}}{3}+2\sqrt{2}\right)}_{\frac{8\sqrt{2}}{3}}\cos\omega_{21}t\,\hat{x}+\underbrace{\left(\frac{2\sqrt{2}}{3}+2\sqrt{2}\right)}_{\frac{8\sqrt{2}}{3}}\sin\omega_{21}t\,\hat{y}$$

$$+\frac{8}{3}\hat{z}\Big\}$$

$$\boxed{\bar{\mu}_m(t)=\frac{-e\hbar}{2m}\left\{\sqrt{2}\cos\omega_{21}t\,\hat{x}+\sqrt{2}\sin\omega_{21}t\,\hat{y}+\hat{z}\right\}}$$