4310, Final Exam December 14, 2015

1. An electron is in the spin state:

$$\chi = A\begin{bmatrix} 3i \\ -2 \end{bmatrix}. \tag{1}$$

2pts (a). Find A.
5 pts (b). Find $\langle S_i \rangle$ for $i = x, y, z$.
5 pts (c). What are the "uncertainties" in S_x, S_y, and S_z? Here, "uncertainty" is taken in the same sense as other observables that we have covered. Check to see if these uncertainties, σ_{S_i}, satisfy $\sigma_{S_i}\sigma_{S_j} \gtreqless \frac{\hbar}{2}|\langle S_k \rangle|$.

(a.) $1 = \langle \chi | \chi \rangle = A^2 \begin{bmatrix} -3i & -2 \end{bmatrix} \begin{bmatrix} 3i \\ -2 \end{bmatrix} = A^2 (9+4) \Rightarrow \boxed{A = \sqrt{\frac{1}{13}}}$

(b.) $\langle S_x \rangle = \langle \chi | \underline{\underline{S}}_x | \chi \rangle = \frac{\hbar}{2}\frac{1}{13}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix}$

$= \frac{\hbar}{26}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 3i \end{bmatrix} = \frac{\hbar}{26}(6i - 6i) = 0$

$\langle S_y \rangle = \frac{\hbar}{26}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar}{26}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 2i \\ -3 \end{bmatrix}$

$= \frac{\hbar}{26}(6+6) = \frac{12\hbar}{26} = \boxed{\frac{6\hbar}{13}}$

$\langle S_z \rangle = \frac{\hbar}{26}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar}{26}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 3i \\ 2 \end{bmatrix} = \boxed{\frac{5\hbar}{26}}$

(c.) $\sigma_i^2 = \langle S_i^2 \rangle - \langle S_i \rangle^2 = \langle \chi | \underline{\underline{S}}_i^2 | \chi \rangle - (\langle \chi | \underline{\underline{S}}_i | \chi \rangle)^2$

$\langle S_x^2 \rangle = \frac{\hbar^2}{4}\frac{1}{13}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix}}_{\begin{bmatrix} -2 \\ 3i \end{bmatrix}} = \frac{\hbar^2}{13 \cdot 4}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar^2}{4}$

$\sigma_x = \boxed{\frac{\hbar}{2}}$

$\langle S_y^2 \rangle = \frac{\hbar^2}{4}\frac{1}{13}\begin{bmatrix} -3i & -2 \end{bmatrix}\underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar^2}{4}\frac{1}{13}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar^2}{4}$

$\sigma_y = \hbar\sqrt{\underbrace{\frac{1}{4} - \frac{36}{169}}} = \boxed{\frac{5\hbar}{26}}$

$\frac{1}{4} - \frac{36}{13^2} = \frac{169-144}{4 \cdot 13^2} = \frac{25}{4 \cdot 13^2} = \frac{5^2}{(2 \cdot 13)^2}$

$\langle S_z^2 \rangle = \frac{\hbar^2}{4}\frac{1}{13}\begin{bmatrix} -3i & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3i \\ -2 \end{bmatrix} = \frac{\hbar^2}{4}$

$\sigma_z = \hbar\sqrt{\frac{1}{4} - \frac{5^2}{26^2}} = \hbar\sqrt{\frac{169-25}{2^2 \cdot 13^2}} = \frac{12\hbar}{26}$

$= \boxed{\frac{6\hbar}{13}}$

(continued)

2. An electron in hydrogen is in the following position and spin state:

$$\Psi(\mathbf{r}, s) = R_{21}\left(\sqrt{\frac{1}{3}}Y_1^0\chi_+ + \sqrt{\frac{2}{3}}Y_1^1\chi_-\right), \qquad (2)$$

where R_{21} and $Y_1^{0,1}$ are defined in the usual way for the hydrogen atom.

5 pts (a). You make a measurement of L^2 on this electron. What values might you get and what are the probabilities of each?

5 pts (b). You make a measurement of L_z on this electron. What values might you get and what are the probabilities of each?

5 pts (c). Same question as (a), except for S^2.

5 pts (d). Same questions as (b), except for S_z.

8 pts (e). What is the probability density? Hint: $R_{21} = \sqrt{\frac{1}{24}}a_0^{-3/2}\frac{r}{a_0}\exp\left(-\frac{r}{2a_0}\right)$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$.

5 pts (f). Find $\langle r\rangle$.

(a.) For both terms, $\ell = 1$ so $L^2\Psi = \hbar^2(1+1)(1)\Psi = \boxed{2\hbar^2}\Psi$ w/ a probability of 1

(b.) For the 1st term, $m=0$ $(L_z=0)$, $P = 1/3$; for the second term, $m=1$ $(L_z=\hbar)$, $P = 2/3$.

(c.) $S^2\Psi = \boxed{\frac{3}{4}\hbar^2}\Psi$ for both terms $(P=1)$

(d.) 1st term: $\frac{\hbar}{2}$, $P = 1/3$; 2nd term: $-\frac{\hbar}{2}$, $P = 2/3$

(e.) Prob. density $= \Psi^*\Psi = |\Psi|^2$

$$= |R_{21}|^2\left(\frac{1}{3}|Y_1^0|^2|\chi_+|^2 + \frac{\sqrt{2}}{3}Y_1^{0*}\chi_+^*Y_1^1\chi_- + \frac{\sqrt{2}}{3}Y_1^{1*}\chi_-^*Y_1^0\chi_+ + \frac{2}{3}|Y_1^1|^2|\chi_-|^2\right)$$

$$= \frac{1}{24a_0^3}\frac{r^2}{a_0^2}e^{-r/a_0}\left[\frac{1}{\cancel{3}}\frac{\cancel{3}}{4\pi}\cos^2\theta|\chi_+|^2 + \frac{\cancel{2}}{\cancel{3}}\frac{\cancel{3}}{\underset{4}{\cancel{8\pi}}}\sin^2\theta|\chi_-|^2\right]$$

where the "mixed" terms go to 0 b/c of the spin-state inner products:

$$\chi_+^*\chi_- = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

$$\chi_-^*\chi_+ = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 0$$

(continued)

3. A particle in a *harmonic oscillator potential* starts out in the following state:

$$\Psi(x,0) = A\left[2\psi_0(x,0) + 3\psi_1(x,0)\right], \tag{3}$$

where ψ_i is the i^{th} harmonic oscillator eigenstate.

2 pts (a). Find A.

6 pts (b). What is $\Psi(x,t)$ and $|\Psi(x,t)|^2$?

9 pts (c). Find $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$.

4 pts (d). Check the position-momentum uncertainty relation.

(a.) $1 = \langle\Psi|\Psi\rangle = A^2(4+9) \Rightarrow A = \sqrt{\frac{1}{13}}$

(b.) $\Psi(x,t) = \sqrt{\frac{1}{13}}\left(2e^{-i\omega_0 t/2} + 3\sqrt{\frac{2m\omega_0}{\hbar}}\,x\,e^{-i\frac{3\omega_0}{2}t}\right)\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega_0}{2\hbar}x^2}$

$$|\Psi(x,t)|^2 = \frac{1}{13}\sqrt{\frac{m\omega_0}{\pi\hbar}}\,e^{\frac{-m\omega_0}{\hbar}x^2}\left[4 + 9\frac{2m\omega_0}{\hbar}x^2 + 6x\sqrt{\frac{2m\omega_0}{\hbar}}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)\right]$$

$$= \frac{1}{13}\sqrt{\frac{m\omega_0}{\pi\hbar}}\,e^{\frac{-m\omega_0}{\hbar}x^2}\left[4 + \frac{18m\omega_0}{\hbar}x^2 + 12x\sqrt{\frac{2m\omega_0}{\hbar}}\cos\omega_0 t\right]$$

or

$$= \frac{1}{13}\left[4\psi_0^2 + 9\psi_1^2 + 12\psi_0\psi_1\cos\omega_0 t\right]$$

Alternative form of $\Psi(x,t) = \sqrt{\frac{1}{13}}\left(2\psi_0 e^{-\frac{i\omega_0}{2}t} + 3\psi_1 e^{-3i\frac{\omega_0}{2}t}\right)$

(c.) $\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\hat{a}_+ + \hat{a}_-\right)$

$$\langle x\rangle = \langle\Psi|\hat{x}|\Psi\rangle = \frac{1}{13}\sqrt{\frac{\hbar}{2m\omega_0}}\Big[\langle 2\psi_0 e^{+\frac{i\omega_0}{2}t} + 3\psi_1 e^{+3i\frac{\omega_0}{2}t}|\hat{a}_+|2\psi_0 e^{-\frac{i\omega_0}{2}t} + 3\psi_1 e^{-3i\frac{\omega_0}{2}t}\rangle + \langle 2\psi_0 e^{+\frac{i\omega_0}{2}t} + 3\psi_1 e^{+\frac{3i\omega_0}{2}t}|\hat{a}_-|2\psi_0 e^{-\frac{i\omega_0}{2}t} + 3\psi_1 e^{-3i\frac{\omega_0}{2}t}\rangle\Big]$$

$$= \frac{1}{13}\sqrt{\frac{\hbar}{2m\omega_0}}\Big[\langle 2\psi_0 e^{+\frac{i\omega_0}{2}t} + 3\psi_1 e^{+3i\frac{\omega_0}{2}t}|2\psi_1 e^{-\frac{i\omega_0}{2}t} + 3\sqrt{2}\psi_2 e^{-3i\frac{\omega_0}{2}t}\rangle + \langle 2\psi_0 e^{+\frac{i\omega_0}{2}t} + 3\psi_1 e^{+\frac{3i\omega_0}{2}t}|0 + 3\psi_0 e^{-\frac{3i\omega_0}{2}t}\rangle\Big]$$

$$= \frac{1}{13}\sqrt{\frac{\hbar}{2m\omega_0}}\left[6\underbrace{\langle\psi_1|\psi_1\rangle}_{1}e^{i\omega_0 t} + 6\underbrace{\langle\psi_0|\psi_0\rangle}_{1}e^{-i\omega_0 t}\right]$$

where $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ so that "mixed terms" are zero.

(continued)

3 pts 4. The electric dipole operator, $\hat{\mu} = q\hat{\mathbf{r}} = q\hat{x}$ (for 1D), couples transitions from ψ_i to ψ_f if the integral $\int_{-\infty}^{\infty} \psi_i^* \hat{\mu} \psi_f$ is non-zero (here, q is the charge). Put slightly differently, if $\psi_i^* \hat{\mu} \psi_f$ is anti-symmetric, then the transition **cannot** occur (*i.e.*, it is forbidden). Between the four main orbital angular momentum states, s ($l = 0$), p ($l = 1$), d ($l = 2$), and f ($l = 3$), which transitions are **dipole allowed**?

$$\left.\begin{matrix} s\ (l=0) \\ d\ (l=2) \end{matrix}\right\}\ \text{symmetric}$$

$$\left.\begin{matrix} p\ (l=1) \\ f\ (l=3) \end{matrix}\right\}\ \text{antisymmetric}$$

<u>Dipole Allowed Transitions</u>

$s \leftrightarrow p$

$s \leftrightarrow f$

$p \leftrightarrow d$

$d \leftrightarrow f$

5 pts

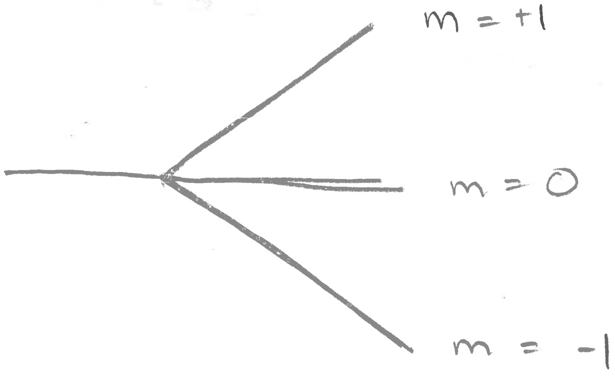
5. The energy splitting of orbital levels for ψ_{nlm} goes as $\Delta E = m\mu_B B$, where μ_B is the Bohr magneton (~57.88 μeV/T for an electron). Ignoring spin, draw the orbital splitting with and without a magnetic field, B, for the s, p, and d orbital states.

$B=0$ $B\neq 0$

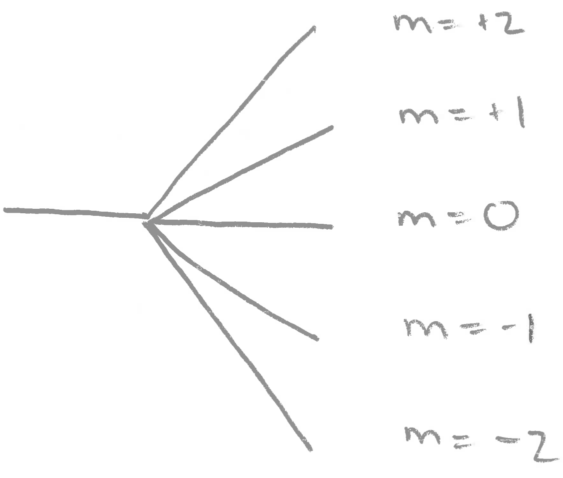
s $(l=0)$

no orbital splitting b/c $l, m = 0$

p $(l=1)$



d $(l=2)$



6. In class, we saw that two eigenstates (wavefunctions) are orthogonal when $\int \psi_a^* \psi_b = 0$.

4 pts (a). Beyond this formula, what does it mathematically mean when we say that two eigenstates (wavefunctions) are orthogonal?

4 pts (b). What does it physically mean when we say that two eigenstates (wavefunctions) are orthogonal?

(a.) The projection of ψ_a onto ψ_b (or vice versa) is 0. Additionally, orthogonal eigenstates have distinct eigenvalues.

(b.) Physically, orthogonality means that the two wavefunctions have zero overlap. Or, put slightly differently, there is no "hopping" or "information" exchanged between the two, non-coupled states. Because ψ_A and ψ_B may correspond to two wavefunctions on two different particles, orthogonality does not imply that we cannot simultaneously measure them; there is no prohibition on that unless the eigenstates are part of the Hilbert space of the same particle.

(a) 8 pts
(b) 3 pts

7. A wavefunction, $\psi(x,0)$, in the infinite square well, of length L, is prepared such that $\psi(x,0) = \frac{1}{\sqrt{2}}[\psi_1 - i\psi_3]$, where ψ_i is the i^{th} infinite square well wavefunction.
(a). Using the explicit form of ψ_i, find $\langle p^2 \rangle$.
(b). What happens to $\langle p^2 \rangle$ if L is tripled and m is halved? What about E_n, where E_n is the energy eigenvalue of the n^{th} level?

(a.)
$$\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right), \quad \psi_3 = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right)$$

$$\langle p^2 \rangle = \langle \psi | \hat{p}^2 | \psi \rangle = -\hbar^2 \int_0^L \frac{1}{2}[\psi_1^* + i\psi_3^*] \frac{d^2}{dx^2}[\psi_1 - i\psi_3]\, dx$$

$$= -\frac{\hbar^2}{2}\frac{2}{L}\int_0^L \left[\sin\frac{\pi x}{L} + i\sin\frac{3\pi x}{L}\right] \underbrace{\frac{d^2}{dx^2}\left[\sin\frac{\pi x}{L} - i\sin\frac{3\pi x}{L}\right]}_{①} dx$$

$$① \quad \frac{d}{dx}\left[\frac{\pi}{L}\cos\frac{\pi x}{L} - i\frac{3\pi}{L}\cos\frac{3\pi x}{L}\right] = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L} + i\frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}$$

$$\langle p^2 \rangle = \frac{\pi^2\hbar^2}{2L^2}\frac{2}{L}\int_0^L \left(\sin\frac{\pi x}{L} + i\sin\frac{3\pi}{L}x\right)\left(\sin\frac{\pi x}{L} - 9i\sin\frac{3\pi x}{L}\right) dx$$

$$= \frac{\pi^2\hbar^2}{2L^2}\left[\int_0^L |\psi_1|^2 dx - 8i\int_0^L \psi_1\psi_3\, dx + 9\int_0^L |\psi_3|^2 dx\right]$$

$= 1$ b/c ψ_1 is normalized

0 by orthogonality

$= 1$ b/c ψ_3 is normalized

$$\boxed{\langle p^2 \rangle = 5\frac{\pi^2\hbar^2}{L^2}}$$

(b.) If L is tripled and m is halved, $\langle p^2 \rangle$ will decrease by a factor of 9 ($\langle p^2 \rangle$ is independent of m, here).

$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, so if L is tripled and m is halved, E_n will decrease by a factor $2/9$.

9 pts 8. At what length scales is it applicable to use quantum mechanics? When do we move into the so-called classical regime?

When the wavelength, λ, of a particle is on the order of or larger than the Bohr radius, a_0, we should use quantum mechanics. This criterion corresponds to particle wavefunctions that have some delocalization beyond the confines of the proton. Classical physics is applicable when the wavefunction can be completely ignored, which occurs when the de Broglie λ is very small.

Quantum: $\lambda = \frac{h}{p} \gtrsim a_0$, where $a_0 = 0.529\ \text{nm}$
$= 5.29 \times 10^{-10}\ \text{m}$

Example:

① $m = 10^{-12}\ \text{kg}$ (1 ng!)
$v = 10^{-3}\ \text{m/s}$
$p = 10^{-15}\ \text{kg} \cdot \text{m/s}$

$$\lambda = \frac{6.626 \times 10^{-34}\ \text{J-s}}{10^{-15}\ \text{kg-m/s}} = 6.626 \times 10^{-19}\ \text{m} < 5.29 \times 10^{-10}\ \text{m}$$

not applicable ⇒ classical regime

② $m_e = 9.1 \times 10^{-31}\ \text{kg}$
$v = 10^{-3}\ \text{m/s}$
$p = 9.1 \times 10^{-34}\ \text{kg} \cdot \text{m/s}$

$$\lambda = \frac{6.626 \times 10^{-34}\ \text{J-s}}{9.1 \times 10^{-34}\ \text{kg-m/s}} \approx \frac{2}{3}\ \text{m} \gg 5.29 \times 10^{-10}\ \text{m}$$

quantum regime

1. (cont.)

$$\sigma_{S_i}\sigma_{S_j} \geq \frac{\hbar}{2}\left|\langle S_k\rangle\right|$$

① $\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}\left|\langle S_z\rangle\right| \Rightarrow \cancel{\frac{\hbar}{2}}\,\cancel{\frac{5\hbar}{26}} \overset{?}{\geq} \cancel{\frac{\hbar}{2}}\,\cancel{\frac{5\hbar}{26}}$

$1 \geq 1$ ✓ (exactly satisfies this relation)

② $\sigma_{S_y}\sigma_{S_z} \geq \frac{\hbar}{2}\left|\langle S_x\rangle\right| \Rightarrow \frac{5\hbar}{26}\,\frac{\hbar}{2}\sqrt{\frac{3}{13}} \overset{?}{\geq} \frac{\hbar}{2}\,0 = 0$ ✓

③ $\sigma_{S_z}\sigma_{S_x} \geq \frac{\hbar}{2}\left|\langle S_y\rangle\right| \Rightarrow \cancel{\frac{6\hbar}{13}}\,\cancel{\frac{\hbar}{2}} \overset{?}{\geq} \cancel{\frac{\hbar}{2}}\,\cancel{\frac{6\hbar}{13}}$

$1 \geq 1$ ✓ (exactly satisfies this relation)

2 (cont.)

$$|\chi_+|^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$|\chi_-|^2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

Prob. density $= \rho(\bar{r}) = \frac{1}{24} \frac{1}{4\pi} \frac{r^2}{a_0^5} e^{-r/a_0} \cancelto{1}{\left[\cos^2\theta + \sin^2\theta\right]}$

$$\boxed{\rho(\bar{r}) = \frac{1}{96\pi} \frac{r^2}{a_0^5} e^{-r/a_0}}$$ ← correct form of R_{21}

$$\boxed{\rho(\bar{r}) = \frac{1}{96\pi} \frac{r^2}{a_0^5} e^{-2r/a_0}}$$ ← using the form of R_{21} provided in the test

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(f.)

$$\langle r \rangle = \int_V \psi^* r \psi \, dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} \frac{1}{96\pi} \frac{r^5}{a_0^5} e^{-r/a_0} dr$$

$$= \frac{1}{24 a_0^5} \int_0^{\infty} r^5 e^{-r/a_0} dr$$

$$\begin{array}{lcl} r^5 & + & e^{-r/a_0} \\ 5r^4 & - & -a_0 e^{-r/a_0} \\ 20r^3 & + & a_0^2 \\ 60r^2 & - & -a_0^3 \\ 120r & + & a_0^4 \\ 120 & - & -a_0^5 \\ 0 & & a_0^6 \end{array}$$

Since only the last term survives, we have:

$$\langle r \rangle = \frac{1}{24 \cancel{a_0^5}} \left[120 a_0^{\cancel{6}\,1} \right]$$

$$\langle r \rangle = \frac{\cancel{8} \cdot \cancel{15}\,5}{\cancel{8} \cdot \cancel{3}} a_0 = \boxed{5 a_0}$$ ← correct form of R_{21}

$$\boxed{\langle r \rangle = \frac{5 a_0}{64}}$$ ← using the form of R_{21} provided in the test

3 (cont.)

$$\langle x\rangle = \frac{1}{13}\sqrt{\frac{\hbar}{2m\omega_0}}\left[6\underbrace{\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)}_{2\cos\omega_0 t}\right]$$

$$\boxed{\langle x\rangle = \frac{6}{13}\sqrt{\frac{2\hbar}{m\omega_0}}\cos\omega_0 t}$$

$$\boxed{\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \frac{-6}{13}\sqrt{2m\hbar\omega_0}\,\sin\omega_0 t}$$

$$\langle x^2\rangle = \frac{1}{13}\frac{\hbar}{2m\omega_0}\langle\psi|\hat{a}_+^2+\hat{a}_+\hat{a}_-+\hat{a}_-\hat{a}_++\hat{a}_-^2|\psi\rangle$$

$$= \frac{1}{13}\frac{\hbar}{2m\omega_0}\left[\langle 2\psi_0 e^{+\frac{i\omega_0}{2}t}+3\psi_1 e^{+\frac{3i\omega_0}{2}t}|\hat{a}_+^2+\hat{a}_+\hat{a}_-+\hat{a}_-\hat{a}_++\hat{a}_-^2|\right.$$
$$\left.2\psi_0 e^{-\frac{i\omega_0}{2}t}+3\psi_1 e^{-\frac{3i\omega_0}{2}t}\rangle\right]$$

$$= \frac{1}{13}\frac{\hbar}{2m\omega_0}\left[\langle 2\psi_0 e^{+\frac{i\omega_0}{2}t}+3\psi_1 e^{+\frac{3i\omega_0}{2}t}|2\sqrt{2}\psi_2 e^{-\frac{i\omega_0}{2}t}+3\sqrt{6}\psi_3 e^{-\frac{3i\omega_0}{2}t}+\right.$$
$$\left.3\psi_1 e^{-\frac{3i\omega_0}{2}t}+2\psi_0 e^{-\frac{i\omega_0}{2}t}+6\psi_1 e^{-\frac{3i\omega_0}{2}t}\rangle\right]$$

$\hat{a}_+^2\psi_0 = \hat{a}_+\psi_1 = \sqrt{2}\psi_2$

$\hat{a}_+^2\psi_1 = \hat{a}_+\sqrt{2}\psi_2 = \sqrt{6}\psi_3$

$\hat{a}_+\hat{a}_-\psi_0 = 0$

$\hat{a}_+\hat{a}_-\psi_1 = \hat{a}_+\psi_0 = \psi_1$

$\hat{a}_-\hat{a}_+\psi_0 = \hat{a}_-\psi_1 = \psi_0$

$\hat{a}_-\hat{a}_+\psi_1 = \hat{a}_-\sqrt{2}\psi_2 = 2\psi_1$

$\hat{a}_-^2\psi_0 = 0$

$\hat{a}_-^2\psi_1 = \hat{a}_-\psi_0 = 0$

$$\langle x^2\rangle = \frac{1}{13}\frac{\hbar}{2m\omega_0}\left[4\overset{1}{\langle\psi_0|\psi_0\rangle}+9\overset{1}{\langle\psi_1|\psi_1\rangle}+18\overset{1}{\langle\psi_1|\psi_1\rangle}\right]$$

$$\boxed{\langle x^2\rangle = \frac{31}{26}\frac{\hbar}{m\omega_0}}$$

$$\hat{p} = i\sqrt{\frac{m\hbar\omega_0}{2}}\left(\hat{a}_+-\hat{a}_-\right)$$

$$\hat{p}^2 = -\frac{m\hbar\omega_0}{2}\left(\hat{a}_+^2-\hat{a}_+\hat{a}_--\hat{a}_-\hat{a}_++\hat{a}_-^2\right)$$

$$\langle \hat{p}^2\rangle = \langle\psi|\hat{a}_+^2|\psi\rangle - \langle\psi|\hat{a}_+\hat{a}_-|\psi\rangle - \langle\psi|\hat{a}_-\hat{a}_+|\psi\rangle + \langle\psi|\hat{a}_-^2|\psi\rangle$$

$$= \frac{-1}{13}\frac{m\hbar\omega_0}{2}\Big[\langle 2\psi_0 e^{\frac{i\omega_0}{2}t} + 3\psi_1 e^{\frac{3i\omega_0}{2}t} | 2\sqrt{2}\psi_2 e^{\frac{-i\omega_0}{2}t} + 3\sqrt{6}\psi_3 e^{\frac{-i3\omega_0}{2}t}\rangle$$

$$-\langle 2\psi_0 e^{\frac{+i\omega_0}{2}t} + 3\psi_1 e^{\frac{3i\omega_0}{2}t} | 3\psi_1 e^{\frac{-3i\omega_0}{2}t}\rangle - \langle 2\psi_0 e^{\frac{i\omega_0}{2}t} + 3\psi_1 e^{\frac{3i\omega_0}{2}t} | 2\psi_0 e^{\frac{-i\omega_0}{2}t} + 6\psi_1 e^{\frac{-i3\omega_0}{2}t}\rangle\Big]$$

$$= \frac{1}{26} m\hbar\omega_0\Big[9\overset{1}{\langle\psi_1|\psi_1\rangle} + 4\overset{1}{\langle\psi_0|\psi_0\rangle} + 18\overset{1}{\langle\psi_1|\psi_1\rangle}\Big]$$

$$\boxed{\langle p^2\rangle = \frac{31}{26}m\hbar\omega_0}$$

(d.) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{31}{26}\frac{\hbar}{m\omega_0} - \frac{36}{13^2}\frac{2\hbar}{m\omega_0}\cos^2\omega_0 t}$

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{31}{26}m\hbar\omega_0 - \frac{36}{13^2}2m\hbar\omega_0\sin^2\omega_0 t}$

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_x = \sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\frac{31}{26} - \frac{72}{169}\cos^2\omega_0 t}$$

$$\sigma_p = \sqrt{m\hbar\omega_0}\sqrt{\frac{31}{26} - \frac{72}{169}\sin^2\omega_0 t}$$

$$\sigma_x\sigma_p = \hbar\sqrt{\underbrace{\left(\frac{31}{26}\right)^2 - \left(\frac{31}{26}\right)\left(\frac{72}{169}\right) + \left(\frac{72}{169}\right)^2\frac{1}{4}\sin^2 2\omega_0 t}}$$

$$\sqrt{a - b\cos^2\omega_0 t}\sqrt{a - b\sin^2\omega_0 t} = a^2 - ab(\overset{1}{\sin^2\omega_0 t + \cos^2\omega_0 t}) + \underbrace{b^2\cos^2\omega_0 t\sin^2\omega_0 t}_{(\sin 2\omega_0 t)^2 = (2\sin\omega_0 t\cos\omega_0 t)^2}$$

$$= a^2 - ab + \frac{b^2}{4}\sin^2 2\omega_0 t$$

$$\sigma_x\sigma_p = \hbar\sqrt{\left(\frac{31}{26}\right)^2 - \left(\frac{31}{26}\right)\left(\frac{72}{169}\right) + \frac{1}{4}\left(\frac{72}{169}\right)^2\underbrace{\sin^2 2\omega_0 t}}$$ always between 0 and 1

since $\frac{72}{169} < 1 < \frac{31}{26}$ the argument under the square root sign is larger than 1. Thus, $\sigma_x\sigma_p > \hbar$ which satisfies $\sigma_x\sigma_p \geq \hbar/2$.