SUBJECT

HW 5 Solns

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For the odd solns, $\psi(-x) = -\psi(x)$

$$\psi(x) = \begin{cases} Fe^{-\kappa x} & x > L \\ C\sin \ell x & x \in (0, L) \\ -\psi(-x) & x < 0 \end{cases}$$

Continuity of $\psi(x)$ @ $x = L$

(1) $Fe^{-\kappa L} = C\sin(\ell L)$

Continuity of $\frac{\partial \psi}{\partial x}$ @ $x = L$

(2) $-\kappa F e^{-\kappa L} = \ell C \cos(\ell L)$

Dividing (2) by (1):

$$-\kappa = \ell \cot(\ell L)$$

where $\ell = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

$$\kappa L = \sqrt{z_0^2 - z^2} \qquad z \equiv \ell L \qquad z_0 \equiv \frac{L}{\hbar}\sqrt{2mV_0}$$

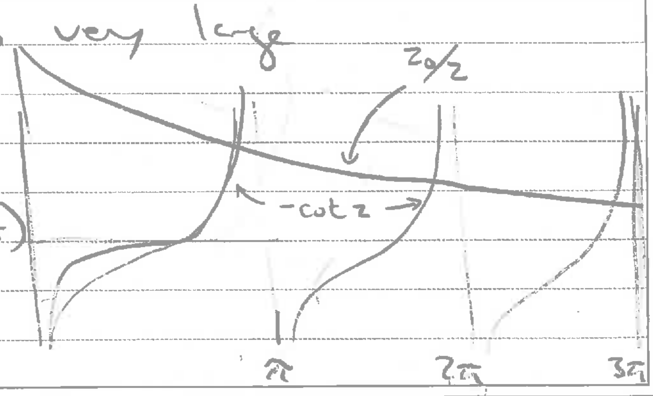
$$\therefore \kappa L = \underbrace{-\ell L}_{z} \cot(\ell L)$$

$$\frac{1}{z}\sqrt{z_0^2 - z^2} = -\cot z$$

$$\boxed{\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} = -\cot z}$$

Wide, deep well: z_0 is very large

$$\frac{z_0}{z} \simeq -\cot z$$



Intersections occur @ (very near) $\pi, 2\pi, 3\pi, \ldots n\pi$

$z_n = \frac{n\pi}{2}$ for n even

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$z \equiv lL \qquad l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

$$z^2/L^2 = l^2 \rightarrow \boxed{\frac{n^2\pi^2}{(2L)^2}\frac{\hbar^2}{2m} = E_n + V_0 \quad \text{for } n \text{ even}}$$

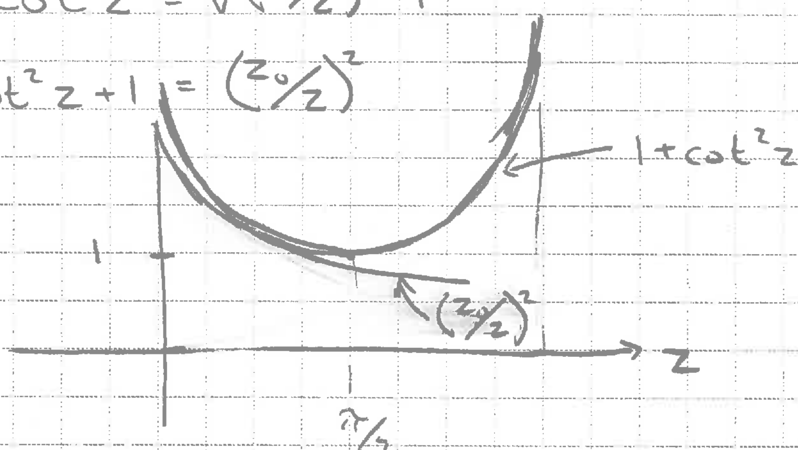
Thus, this acts to fill the half of the infinite well limiting case.

<u>Shallow, narrow well</u>: z_0 is small

We can construct a scenario whereby z_0 is so small that no intersection occurs; that is, no bound state for the odd soln.

$$-\cot z = \sqrt{(z_0/z)^2 - 1}$$

$$\cot^2 z + 1 = (z_0/z)^2$$



If $(z_0/z)^2 < 1$ @ $z = \pi/2$, then no bnd state. This is accomplished when $z_0 < \pi/2$

$$z_0 < \pi/2$$

$$\frac{L}{\hbar}\sqrt{2mV_0} < \pi/2$$

$$V_0 < \frac{\hbar^2\pi^2}{4L^2 2m} = \frac{\pi^2\hbar^2}{8mL^2}$$

When this condition is satisfied, no odd bnd state.

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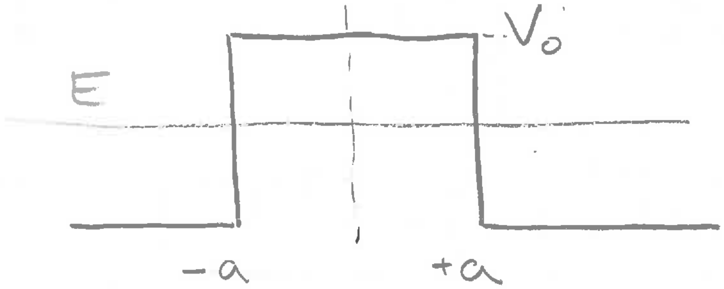
$E < V_0$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x(-\infty, -a) \\ Ce^{Kx} + De^{-Kx} & x(-a, a) \\ Fe^{ikx} & x(a, \infty) \end{cases}$$



$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Continuity of ψ @ $x = -a$:

$$① \; Ae^{-ika} + Be^{ika} = Ce^{-Ka} + De^{+Ka}$$

Continuity of $\frac{d\psi}{dx}$ @ $x = -a$

$$ik(Ae^{-ika} - Be^{ika}) = K(Ce^{-Ka} - De^{+Ka})$$

$$② \; Ae^{-ika} - Be^{ika} = \frac{K}{ik}(Ce^{-Ka} - De^{+Ka})$$

Adding ① and ②: $2Ae^{-ika} = \left(1 + \frac{K}{ik}\right)Ce^{-Ka} + \left(1 - \frac{K}{ik}\right)De^{+Ka}$

$$☆ \quad 2Ae^{-ika} = \left(1 - \frac{iK}{k}\right)Ce^{-Ka} + \left(1 + \frac{iK}{k}\right)De^{+Ka}$$

Continuity of ψ @ $x = +a$:

$$③ \; Ce^{Ka} + De^{-Ka} = Fe^{ika}$$

Continuity of $\frac{d\psi}{dx}$ @ $x = +a$:

$$K(Ce^{Ka} - De^{-Ka}) = ikFe^{ika}$$

$$④ \; (Ce^{Ka} - De^{-Ka}) = \frac{ik}{K}Fe^{ika}$$

Adding ③ & ④: $2Ce^{Ka} = \left(1 + \frac{ik}{K}\right)Fe^{ika}$ ☆☆

Subtracting ③ & ④: $2De^{-Ka} = \left(1 - \frac{ik}{K}\right)Fe^{ika}$ ☆☆☆

Putting $\star\star$ and $\star\star\star$ into $\star$:

$$2Ae^{-ika} = \frac{1}{2}e^{-2Ka}\left(1-\frac{iK}{k}\right)\left(1+\frac{ik}{K}\right)Fe^{ika} + \frac{1}{2}e^{+2Ka}\left(1+\frac{iK}{k}\right)\left(1-\frac{ik}{K}\right)Fe^{ika}$$

$$= \frac{1}{2}Fe^{ika}\left(\left[1+i\left(\frac{k}{K}-\frac{K}{k}\right)+1\right]e^{-2Ka} + \left[1+i\left(\frac{K}{k}-\frac{k}{K}\right)+1\right]e^{+2Ka}\right)$$

$$= \frac{1}{2}Fe^{ika}\left[2\left(e^{+2Ka}+e^{-2Ka}\right) + \frac{i}{kK}\left[(k^2-K^2)e^{-2Ka} + (K^2-k^2)e^{+2Ka}\right]\right]$$

$$= \frac{1}{2}Fe^{ika}\left[2\underbrace{\left(e^{+2Ka}+e^{-2Ka}\right)}_{2\cosh(2Ka)} + \frac{i(K^2-k^2)}{kK}\underbrace{\left(e^{+2Ka}-e^{-2Ka}\right)}_{2\sinh(2Ka)}\right]$$

$$2Ae^{-ika} = Fe^{ika}\left[2\cosh(2Ka) + \frac{i(K^2-k^2)}{kK}\sinh(2Ka)\right]$$

$$\frac{A}{F} = e^{2ika}\left[\cosh(2Ka) + \frac{i(K^2-k^2)}{2kK}\sinh(2Ka)\right]$$

$$T^{-1} = \left|\frac{A}{F}\right|^2 = \cosh^2(2Ka) + \left[\frac{(K^2-k^2)}{2kK}\right]^2\sinh^2(2Ka)$$

$$\cosh^2\theta = 1+\sinh^2\theta$$

$$T^{-1} = 1 + \left[1+\left(\frac{K^2-k^2}{2kK}\right)^2\right]\sinh^2(2Ka)$$

$$\frac{(2kK)^2+K^4-2k^2K^2+k^2}{4k^2K^2} = \frac{K^4+2k^2K^2+k^2}{4k^2K^2} = \frac{(K^2+k^2)^2}{4k^2K^2}$$

$$\frac{\left[\cancel{2m}(V_0-E)+\cancel{2m}E\right]^2}{4(\cancel{2m}[V_0-E])(\cancel{2m}E)} = \frac{V_0^2}{4E(V_0-E)}$$

$$\boxed{T^{-1} = 1 + \left[\frac{V_0^2}{4E(V_0-E)}\right]\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}$$

$E = V_0$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x(-\infty, -a) \\ C + Dx & x(-a, a) \\ Fe^{ikx} & x(a, \infty) \end{cases} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

For the middle region: $\frac{d^2\psi}{dx^2} + \cancel{V_0\psi} = \cancel{E\psi}$ For $E = V_0$

$$\frac{d^2\psi}{dx^2} = 0$$

$$\psi = C + Dx$$

Continuity of ψ @ $x = -a$

$$Ae^{-ika} + Be^{ika} = C - Da$$

Continuity of ψ @ $x = +a$

$$C + Da = Fe^{+ika}$$

Continuity of $\frac{d\psi}{dx}$ @ $x = -a$

$$ik(Ae^{-ika} - Be^{+ika}) = D$$

Continuity of $\frac{d\psi}{dx}$ @ $x = +a$

$$D = ikFe^{ika}$$

② $Ae^{-ika} - Be^{+ika} = Fe^{ika}$

$$Ae^{-ika} + Be^{ika} = Fe^{ika} - 2Da$$

① $Ae^{-ika} + Be^{ika} = Fe^{ika}(1 - 2ika)$

Adding ① + ②:

$$\cancel{2}Ae^{-ika} = \cancel{2}Fe^{ika}(1 - ika)$$

$$\frac{A}{F} = e^{2ika}(1 - ika)$$

$$T^{-1} = \left|\frac{A}{F}\right|^2 = (1-ika)(1+ika) = 1+(ka)^2$$

$$\boxed{T^{-1} = 1 + \frac{2mEa^2}{\hbar^2}}$$

$E > V_0$

Same case as the book, except w/ $+V_0$, instead of $-V_0$:

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E-V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right)}$$

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$$V(x) = \begin{cases} 0 & x \le 0 \\ V_0 & x > 0 \end{cases}$$

(a.)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \quad k = \frac{\sqrt{2mE}}{\hbar} \\ Fe^{-Kx} & x > 0 \quad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \end{cases}$$

Continuity of ψ @ $x = 0$

$$A + B = F$$

Continuity of $\frac{d\psi}{dx}$ @ $x = 0$

$$ik(A - B) = -KF$$

$$ik(A - B) = -K(A + B)$$

$$(ik + K)A = B(ik - K)$$

$$\frac{(ik + K)}{(ik - K)} = \frac{B}{A}$$

$$R = \frac{|B|^2}{|A|^2} = \frac{(ik + K)(-ik + K)}{(ik - K)(-ik - K)} = \frac{k^2 + K^2}{k^2 + K^2} = 1$$

When $E < V_0$ for the infinite step potential, all of the incident wave is reflected.

(b.) First we need to re-write our ψ:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \quad k = \frac{\sqrt{2mE}}{\hbar} \\ Fe^{ilx} & x > 0 \quad l = \frac{\sqrt{2m(E - V_0)}}{\hbar} \end{cases}$$

① $A + B = F$ $\qquad \psi(0)$

② $ik(A - B) = ilF$ $\qquad \left.\frac{d\psi}{dx}\right|_{x=0}$

Solving for B:

$$k(A-B) = l(A+B)$$

$$\bigstar \quad \frac{k-l}{k+l} A = B$$

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{k-l}{k+l}\right)^2 = \frac{k^2 - 2kl + l^2}{k^2 + 2kl + l^2}$$

$$R = \frac{2\cancel{m}E - 2\cancel{m}\sqrt{E(E-V_0)} + 2\cancel{m}(E-V_0)}{2\cancel{m}E + 2\cancel{m}\sqrt{E(E-V_0)} + 2\cancel{m}(E-V_0)}$$

$$= \frac{2E - V_0 - \sqrt{E(E-V_0)}}{2E - V_0 + \sqrt{E(E-V_0)}}$$

(c.)

Free particle probability current (incident):

Method 1

$$J_i = \frac{\hbar k}{m}|A|^2$$

v_0 $\quad$ v_t

"Transmitted probability current", $J_t = \frac{\hbar l}{m}|F|^2$

$$T = \frac{J_t}{J_i} = \frac{l}{k}\left|\frac{F}{A}\right|^2 = \sqrt{\frac{E-V_0}{E}}\left|\frac{F}{A}\right|^2$$

$$T = \frac{P_{trans}}{P_{incident}} = \frac{v_t|F|^2}{v_0|A|^2}$$ } same statement as J_t/J_i

Method 2

$$v_0 = \sqrt{\frac{2E}{m}} \qquad v_t = \sqrt{\frac{2(E-V_0)}{m}}$$

As we saw in class, $T = 0$ for $E < V_0$.

(d.) For $E > V_0$

$F = A + B$ (continuity @ $x=0$) $\quad \Big\}$ $F = A\left[\frac{k+l+k-l}{k+l}\right]$

From ★ above: $\left(\frac{k-l}{k+l}\right)A = B$ $\quad$ $F = A\left(\frac{2k}{k+l}\right)$

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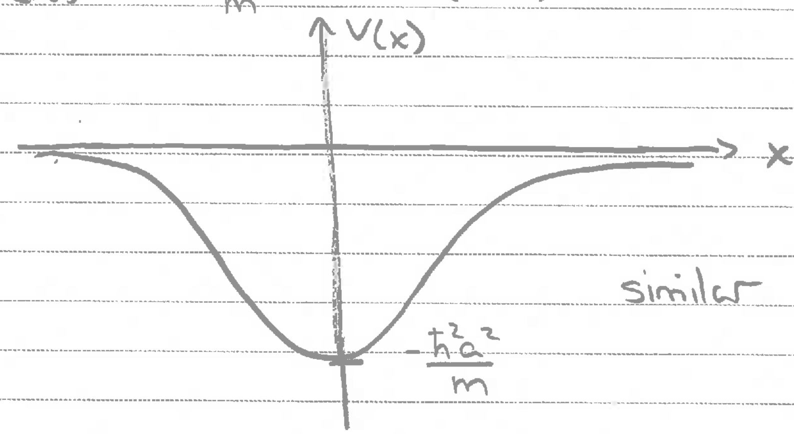
$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \frac{4\cancel{k^2}^{\,k}}{(k+l)^2}\frac{l}{\cancel{k}} = \frac{4kl}{(k+l)^2} = \frac{\frac{4\sqrt{2mE}}{\hbar}\frac{\sqrt{2m(E-V_0)}}{\hbar}}{\frac{1}{\hbar^2}\left(\sqrt{2mE}+\sqrt{2m(E-V_0)}\right)^2}$$

$$= 4\left[\frac{\sqrt{E(E-V_0)}}{E+(E-V_0)+2\sqrt{E(E-V_0)}}\right] = \frac{4\sqrt{E(E-V_0)}}{2E-V_0+2\sqrt{E(E-V_0)}}$$

$$T+R = \frac{4kl}{(k+l)^2} + \underbrace{\left(\frac{k-l}{k+l}\right)^2}_{\text{from part (b.)}} = \frac{k^2-2kl+l^2+4kl}{k^2+2kl+l^2} = \frac{k^2+2kl+l^2}{k^2+2kl+l^2} = 1 \checkmark$$

2.51 $V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)$

(a.)



similar to a Gaussian

(b.) $H\psi_0 = \frac{-\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V(x)\psi_0$, where $\psi_0 = A\operatorname{sech}(ax)$

$$\frac{d\psi_0}{dx} = -Aa\tanh(ax)\operatorname{sech}(ax)$$

$$\frac{d}{dx}\operatorname{sech}(ax) = -a\tanh(ax)\operatorname{sech}(ax)$$

$$\frac{d}{dx}\tanh(ax) = a\operatorname{sech}^2(ax)$$

$$\frac{d^2\psi_0}{dx^2} = -Aa\left[a\operatorname{sech}^3(ax) - a\tanh^2(ax)\operatorname{sech}(ax)\right]$$

$$= Aa^2\operatorname{sech}(ax)\left[\tanh^2(ax) - \operatorname{sech}^2(ax)\right]$$

$$H\psi_0 = -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V(x)\psi_0 = -\frac{Aa^2\hbar^2}{2m}\left[\operatorname{sech}(ax)\tanh^2(ax) - \operatorname{sech}^3(ax)\right] - \frac{A\hbar^2a^2}{m}\operatorname{sech}^3(ax)$$

$$= \frac{-Aa^2\hbar^2}{2m}\left[\operatorname{sech}(ax)\tanh^2(ax) + \operatorname{sech}^3(ax)\right]$$

$$= -\frac{\hbar^2a^2}{2m}A\operatorname{sech}(ax)\left[\tanh^2(ax) + \operatorname{sech}^2(ax)\right]$$

$$\frac{\sinh^2(ax)}{\cosh^2(ax)} + \frac{1}{\cosh^2(ax)} = \frac{1+\sinh^2(ax)}{\cosh^2(ax)}$$

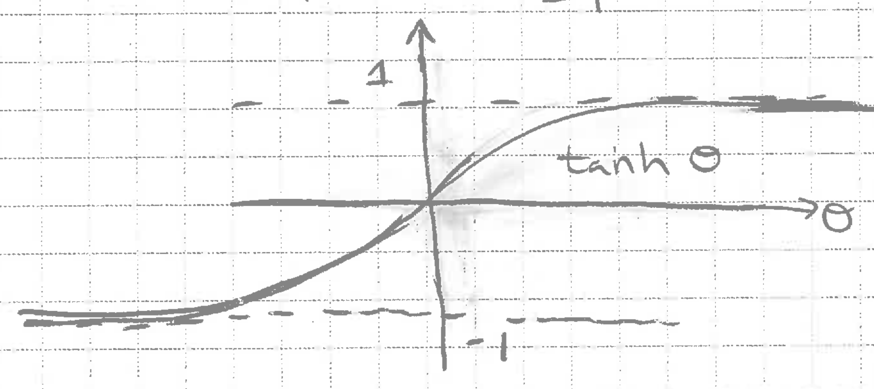
$$\frac{1+\sinh^2(ax)}{\cosh^2(ax)} = \frac{4+(e^{ax}-e^{-ax})^2}{(e^{ax}+e^{-ax})^2} = \frac{e^{2ax}+2+e^{-2ax}}{e^{2ax}+2+e^{-2ax}} = 1$$

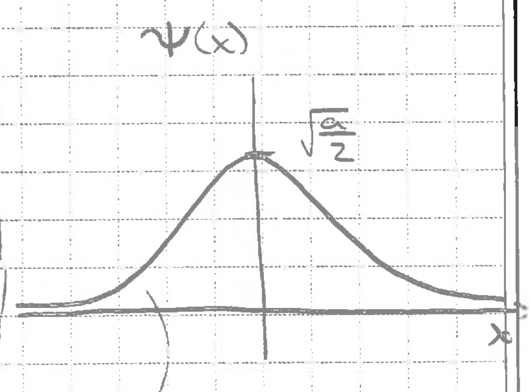
$$= \frac{-\hbar^2a^2}{2m}\underbrace{A\operatorname{sech}(ax)}_{\psi_0}$$

$$\therefore \boxed{H\psi_0 = \frac{-\hbar^2a^2}{2m}\psi_0} \qquad \boxed{E = \frac{-\hbar^2a^2}{2m}}$$

$$1 = \psi_0^*\psi_0 = |A|^2\int_{-\infty}^{\infty}\operatorname{sech}^2(ax)\,dx = A^2\left[\frac{1}{a}\tanh(ax)\Big|_{-\infty}^{\infty}\right]$$

$$1 = \frac{A^2}{a}\left[\underbrace{\tanh(\infty)}_{1} - \underbrace{\tanh(-\infty)}_{-1}\right] \Rightarrow 1 = \frac{2}{a}A^2 \Rightarrow \boxed{A = \sqrt{\frac{a}{2}}}$$





$$\psi_0 = \sqrt{\frac{a}{2}}\operatorname{sech}(ax)$$

(c.) $\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik + a}\right)e^{ikx}$

$$\psi_k = \left(\frac{A}{ik+a}\right)\left[ike^{ikx} - a\tanh(ax)e^{ikx}\right]$$

$$\frac{d\psi_k}{dx} = \left(\frac{A}{ik+a}\right)\left[-k^2e^{ikx} - a\left(ik\tanh(ax)e^{ikx} + a\,\mathrm{sech}^2(ax)e^{ikx}\right)\right]$$

$$\frac{d^2\psi_k}{dx^2} = \left(\frac{A}{ik+a}\right)\Big[-ik^3e^{ikx} + ak^2\tanh(ax)e^{ikx} - ika^2\mathrm{sech}^2(ax)e^{ikx} +$$
$$2a^3\tanh(ax)\mathrm{sech}^2(ax)e^{ikx} - ika^2\mathrm{sech}^2(ax)e^{ikx}\Big]$$

$$\frac{d^2\psi_k}{dx^2} = \left(\frac{A}{ik+a}\right)e^{ikx}\Big[-ik^3 + ak^2\tanh(ax) - 2ika^2\mathrm{sech}^2(ax) +$$
$$2a^3\tanh(ax)\mathrm{sech}^2(ax)\Big]$$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} + V(x)\psi_k = \left(\frac{A}{ik+a}\right)e^{ikx}\Big[\frac{-\hbar^2}{2m}\Big(-ik^3 + ak^2\tanh(ax) - 2ika^2\mathrm{sech}^2(ax) +$$
$$2a^3\tanh(ax)\mathrm{sech}^2(ax)\Big) - \frac{\hbar^2a^2}{m}\mathrm{sech}^2(ax)\left(ik - a\tanh(ax)\right)\Big]$$

$$= \left(\frac{A}{ik+a}\right)e^{ikx}\left(\frac{-\hbar^2}{2m}\right)\Big[-ik^3 + ak^2\tanh(ax) - 2ia^2k\,\mathrm{sech}^2(ax) +$$
$$2a^3\tanh(ax)\mathrm{sech}^2(ax) + 2ia^2k\,\mathrm{sech}^2(ax) - 2a^3\tanh(ax)\mathrm{sech}^2(ax)\Big]$$

$$= \left(\frac{A}{ik+a}\right)e^{ikx}\left(-\frac{\hbar^2k^2}{2m}\right)\left[a\tanh(ax) - ik\right]$$

$$= \underbrace{\left(\frac{\hbar^2k^2}{2m}\right)}_{E_k}\underbrace{\left(\frac{A}{ik+a}\right)\left[ik - a\tanh(ax)\right]e^{ikx}}_{\psi_k(x)}$$

$$= E_k\psi_k(x) \qquad \text{QED}$$

As $x \to \infty$, $\boxed{\psi_k(x\to\infty) \to A\left(\frac{ik-a}{ik+a}\right)e^{ikx}}$, since $\tanh(\infty) \to 1$.

For large, neg. x (incoming wave) $\psi_k \approx A\left(\frac{ik+a}{ik+a}\right)e^{ikx} = Ae^{ikx}$

$$T = \frac{\text{outgoing wave prob.}}{\text{incoming wave prob.}} = \frac{\left|A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right|^2}{\left|A e^{ikx}\right|^2}$$

$$T = \left|\frac{ik-a}{ik+a}\right|^2 = \frac{ik-a}{ik+a}\,\frac{-ik-a}{-ik+a} = \frac{(-1)(ik-a)(ik+a)}{(-1)(ik+a)(ik-a)} = 1$$

$R+T=1$. If $T=1$, then $R=0$.