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4.28

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle S_x \rangle = \chi^\dagger \underline{\underline{S}}_x \chi = \frac{\hbar}{2}(a^* \ b^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^* \ b^*)\begin{pmatrix} b \\ a \end{pmatrix}$$

$$= \frac{\hbar}{2}(a^* b + a b^*)$$

$$\langle S_y \rangle = \chi^\dagger \underline{\underline{S}}_y \chi = \frac{\hbar}{2}(a^* \ b^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^* \ b^*)\begin{pmatrix} -ib \\ ia \end{pmatrix}$$

$$= \frac{\hbar}{2}(-ia^* b + iab^*) = \frac{-i\hbar}{2}(a^* b - ab^*)$$

$$\langle S_z \rangle = \chi^\dagger \underline{\underline{S}}_z \chi = \frac{\hbar}{2}(a^* \ b^*)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^* \ b^*)\begin{pmatrix} a \\ -b \end{pmatrix}$$

$$= \frac{\hbar}{2}(|a|^2 - |b|^2)$$

$$S_x^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle S_x^2 \rangle = \langle \chi | \hat{S}_x^2 | \chi \rangle = \frac{\hbar^2}{4}(a^* \ b^*)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar^2}{4}(a^* \ b^*)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\underbrace{(|a|^2 + |b|^2)}_{1} = \frac{\hbar^2}{4}$$

$$S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \underbrace{\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{same as before}}$$

so: $\langle S_y^2 \rangle = \frac{\hbar^2}{4}$

$$S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

same as S_x^2, so $\langle S_z^2 \rangle = \frac{\hbar^2}{4}$

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{3\hbar^2}{4} \quad ✓$$

$S^2|\chi\rangle = \hbar^2 s(s+1)|\chi\rangle$. Here, $s = 1/2$ so

$\frac{\hbar^2}{2}\left(\frac{1}{2}+1\right) = \frac{3\hbar^2}{4} \rightarrow S^2|\chi\rangle = \frac{3\hbar^2}{4}|\chi\rangle$ or $\langle S^2 \rangle = \langle \chi | \hat{S}^2 | \chi \rangle$

$\langle S^2 \rangle = \frac{3\hbar^2}{4}$

4.33

$$\bar{B} = B_0 \cos(\omega t)\hat{k}$$

$$\underline{H} = -\gamma \bar{B} \cdot \underline{S}$$

(a.) $\underline{H} = -\gamma B_0 \cos(\omega t) \underline{S}_z$

$$\boxed{\underline{H} = -\frac{\gamma B_0 \hbar \cos(\omega t)}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(b.)

$$\chi(0) = \chi_+^{(x)} = |\uparrow\rangle_x = \begin{bmatrix} \alpha(0) \\ \beta(0) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenspinor of S_x for $|\uparrow\rangle_x$

$$\chi(t) = \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}$$

$$\underline{H}\chi = i\hbar \frac{\partial \chi}{\partial t} \Rightarrow -\frac{\gamma B_0 \hbar \cos(\omega t)}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = i\hbar \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

$$-\frac{\gamma B_0 \hbar \cos(\omega t)}{2} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = i\hbar \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

$$\tfrac{1}{2} i\gamma B_0 \cos(\omega t)\alpha = \dot{\alpha} \longrightarrow \frac{i\gamma B_0 \cos(\omega t)}{2} dt = \frac{d\alpha}{\alpha} \quad (1)$$

$$-\tfrac{1}{2} i\gamma B_0 \cos(\omega t)\beta = \dot{\beta} \longrightarrow \frac{-i\gamma B_0 \cos(\omega t) dt}{2} = \frac{d\alpha}{\alpha} \quad (2)$$

(1) $\ln\alpha + C = \frac{i\gamma B_0}{2\omega} \sin \omega t + C'$

$$\alpha = C_0 \exp\left(\frac{i\gamma B_0}{2\omega} \sin(\omega t)\right), \text{ where } C_0 = \alpha(0) = \frac{1}{\sqrt{2}}$$

(2) $\ln\beta + D = \frac{-i\gamma B_0}{2\omega} \sin(\omega t) + D'$

$$\beta = D_0 \exp\left(\frac{-i\gamma B_0}{2\omega} \sin(\omega t)\right), \text{ where } D_0 = \beta(0) = \frac{1}{\sqrt{2}}$$

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \exp\left(\frac{i\gamma B_0}{2\omega} \sin \omega t\right) \\ \exp\left(\frac{-i\gamma B_0}{2\omega} \sin \omega t\right) \end{bmatrix}}$$

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(c.)

$$\chi = c_+^{(x)}\chi_+^{(x)} + c_-^{(x)}\chi_-^{(x)}$$

$$[\chi_-^{(x)}]^\dagger \chi = c_+^{(x)}\underbrace{[\chi_-^{(x)}]^\dagger \chi_+^{(x)}}_{=0} + c_-^{(x)}\underbrace{[\chi_-^{(x)}]^\dagger \chi_-^{(x)}}_{\frac{1}{2}[1\ -1]\begin{bmatrix}1\\-1\end{bmatrix} = 1}$$

$$\frac{1}{\sqrt{2}}[1\ \ -1]\frac{1}{\sqrt{2}}\begin{bmatrix}\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\\ \exp\left(\frac{-i\gamma B_0}{2\omega}\sin\omega t\right)\end{bmatrix} = c_-$$

$$\underbrace{\frac{1}{2}\left[\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(\frac{-i\gamma B_0}{2\omega}\sin\omega t\right)\right]}_{i\sin\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)} = c_-$$

$$\boxed{\text{Probability} = |c_-|^2 = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)}$$

(d.) A complete spin flip is achieved when $|c_-|^2 = 1$, which occurs when $\sin^2(\pi/2) = 1$. Therefore, taking $\sin\omega t = 1$

$$\frac{\gamma B_0}{2\omega} = \pi/2$$

$$\boxed{B_0 = \frac{\pi\omega}{\gamma}}$$

We take $\sin\omega t = 1$ since we are only concerned w/ the minimum value of B_0. Thus, we can always wait until $\sin\omega t$ reaches 1 for our condition of $B_0 = \frac{\pi\omega}{\gamma}$ to be satisfied.

4.34

(a.) $S_- = S_-^{(1)} + S_-^{(2)}$

$$S_-|1\,0\rangle = S_-\left(\tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\right) = \frac{1}{\sqrt{2}}\left[\underbrace{(S_-^{(1)}\uparrow)}_{\hbar\downarrow}\downarrow + \downarrow\underbrace{(S_-^{(2)}\uparrow)}_{\hbar\downarrow}\right]$$

$$S_-|1\,0\rangle = \frac{\hbar}{\sqrt{2}}(\downarrow\downarrow + \downarrow\downarrow) = \hbar\sqrt{2}\,|1\,{-1}\rangle \checkmark$$

$$|1\,{-1}\rangle + |1\,{-1}\rangle = 2|1\,{-1}\rangle$$

(b.) $S_\pm|0\,0\rangle = S_\pm \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

$S_+ = S_+^{(1)} + S_+^{(2)}$

S_+: $\quad \frac{1}{\sqrt{2}}\left[(S_+^{(1)}\uparrow)\downarrow - \downarrow S_+^{(2)}\uparrow\right] = 0$, where $S_+^i \uparrow = 0$ ✓

S_-: $\quad \frac{1}{\sqrt{2}}\left[(S_-^{(1)}\uparrow)\downarrow - \downarrow S_-^{(2)}\uparrow\right] = \frac{1}{\sqrt{2}}\left[\hbar\downarrow\downarrow - \downarrow\hbar\downarrow\right]$

$$= \frac{\hbar}{\sqrt{2}}\left[\downarrow\downarrow - \downarrow\downarrow\right] = 0 \checkmark$$

(c.)

$$S^2 = [S^{(1)}]^2 + [S^{(2)}]^2 + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$

$$S^2|1\,1\rangle = [S^{(1)}]^2|1\,1\rangle + [S^{(2)}]^2|1\,1\rangle + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}|1\,1\rangle$$

$$= (S^{(1)2}\uparrow)\uparrow + \uparrow(S^{(2)2}\uparrow) + 2\left[S_x^{(1)}\uparrow S_x^{(2)}\uparrow + S_y^{(1)}\uparrow S_y^{(2)}\uparrow + S_z^{(1)}\uparrow S_z^{(2)}\uparrow\right]$$

$$= \frac{3\hbar^2}{4}\uparrow\uparrow + \frac{3\hbar^2}{4}\uparrow\uparrow + 2\left[\frac{\hbar}{2}\downarrow\frac{\hbar}{2}\downarrow + \frac{i\hbar}{2}\downarrow\frac{i\hbar}{2}\downarrow + \frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow\right]$$

$$= \frac{3\hbar^2}{2}\uparrow\uparrow + \frac{\hbar^2}{2}\left[\downarrow\downarrow - \downarrow\downarrow + \uparrow\uparrow\right]$$

$$= 2\hbar^2\uparrow\uparrow$$

$S^2|1\,1\rangle = 2\hbar^2|1\,1\rangle$, which agrees w/ $S^2|1\,1\rangle = \hbar^2 s(s+1)|1\,1\rangle$ for a $s=1$ system.

$$S^2|1\,{-1}\rangle = (S^{(1)2}\downarrow)\downarrow + \downarrow[S^{(2)2}]\downarrow + 2\left[S_x^{(1)}\downarrow S_x^{(2)}\downarrow + S_y^{(1)}\downarrow S_y^{(2)}\downarrow + S_z^{(1)}\downarrow S_z^{(2)}\downarrow\right]$$

$$= \frac{3\hbar^2}{4}\downarrow\downarrow + \frac{3\hbar^2}{4}\downarrow\downarrow + 2\left[\frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow + \left(\frac{-i\hbar}{2}\uparrow\right)\left(\frac{-i\hbar}{2}\uparrow\right) + \left(\frac{-\hbar}{2}\downarrow\right)\left(\frac{-\hbar}{2}\downarrow\right)\right]$$

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$$S^2|1\ -1\rangle = \frac{3\hbar^2}{2}\downarrow\downarrow + 2\frac{\hbar^2}{4}[\uparrow\uparrow - \uparrow\uparrow + \downarrow\downarrow]$$

$$= 2\hbar^2\downarrow\downarrow = 2\hbar^2|1\ -1\rangle$$

Again, we expect $\hbar^2 s(s+1) = 2\hbar^2$ as the eigenvalue for a $s=1$ system.

4.49

$$\chi = A\begin{bmatrix}1-2i\\2\end{bmatrix}$$

(a.) $\langle\chi|\chi\rangle = 1$

$$A^2[1+2i\ \ 2]\begin{bmatrix}1-2i\\2\end{bmatrix} = A^2(5+4) = 1$$

$$A = 1/3$$

(b.)

$$\chi = \frac{1}{3}\begin{bmatrix}1-2i\\2\end{bmatrix}$$

$$\chi_+^\dagger\chi = \frac{1}{3}[1\ \ 0]\begin{bmatrix}1-2i\\2\end{bmatrix} = \frac{1}{3}(1-2i) = c_\uparrow$$

$|c_\uparrow|^2 = \frac{1}{9}(1+2i)(1-2i) = 5/9$ = probability of getting $\hbar/2$ (spin up) in the z-direction.

$$\chi_-^\dagger\chi = \frac{1}{3}[0\ \ 1]\begin{bmatrix}1-2i\\2\end{bmatrix} = \frac{2}{3} = c_\downarrow$$

$|c_\downarrow|^2 = \frac{4}{9}$; probability of getting $-\hbar/2$ (spin down) in the z-direction

$$\langle S_z\rangle = \chi^\dagger S_z\chi = \frac{\hbar}{2}\frac{1}{9}[1-2i\ \ 2]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1-2i\\2\end{bmatrix}$$

$$= \frac{\hbar}{18}[1+2i\ \ 2]\begin{bmatrix}1-2i\\-2\end{bmatrix} = \frac{\hbar}{18}(1+4-4) = \frac{\hbar}{18}$$

or

$$\langle S_z\rangle = \underbrace{\frac{\hbar}{2}\left(\frac{5}{9}\right)}_{\text{prob. of getting } \hbar/2} + \underbrace{\left(-\frac{\hbar}{2}\right)\left(\frac{4}{9}\right)}_{\text{prob. of getting } -\hbar/2} = \frac{\hbar}{18}$$

(c.)

Same procedure as before:

$$c_+^{(x)} = \left(\chi_+^{(x)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{3}\begin{bmatrix}1-2i \\ 2\end{bmatrix}$$

$$= \frac{1}{3\sqrt{2}}(1-2i+2) = \frac{1}{3\sqrt{2}}(3-2i)$$

$$|c_+^{(x)}|^2 = \frac{1}{18}(3-2i)(3+2i) = \frac{13}{18}$$ prob. of getting $\hbar/2$ $(\uparrow_x)$ in x-direction

$$c_-^{(y)} = \left(\chi_-^{(x)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\frac{1}{3}\begin{bmatrix}1-2i \\ 2\end{bmatrix} = \frac{1}{3\sqrt{2}}(1-2i-2)$$

$$= \frac{-1}{3\sqrt{2}}(1+2i)$$

$$|c_-^{(x)}|^2 = \frac{1}{18}(1+2i)(1-2i) = \frac{5}{18}$$ prob. of getting $-\hbar/2$ $(\downarrow_x)$ in x-direction

$$\langle S_x \rangle = \frac{13}{18}\frac{\hbar}{2} + \frac{5}{18}\left(-\frac{\hbar}{2}\right) = \frac{8}{18}\frac{\hbar}{2} = \frac{2\hbar}{9}$$

(d.)

$$c_+^{(y)} = \left(\chi_+^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\frac{1}{3}\begin{bmatrix}1-2i \\ 2\end{bmatrix} = \frac{1}{3\sqrt{2}}\left[1-2i-2i\right]$$

$$= \frac{1}{3\sqrt{2}}(1-4i)$$

$$|c_+^{(y)}|^2 = \frac{1}{18}(1+16) = \frac{17}{18}$$ prob. of getting $\hbar/2$ in the y-direction

$$c_-^{(y)} = \left(\chi_-^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & +i\end{bmatrix}\frac{1}{3}\begin{bmatrix}1-2i \\ 2\end{bmatrix} = \frac{1}{3\sqrt{2}}\left[1-2i+2i\right]$$

$$c_-^{(y)} = \frac{1}{3\sqrt{2}}$$

$$\boxed{|c_-^{(y)}|^2 = \frac{1}{18} \quad \text{prob. of getting } -\hbar/2 \text{ in the y-direction}}$$

$\frac{16}{18}$

$$\langle S_y \rangle = \frac{\hbar}{2}\left(\frac{17}{18}\right) + \frac{-\hbar}{2}\left(\frac{1}{18}\right) = \frac{4\hbar}{9}$$