HW #2 Solutions

1.

(a.)

$$x_0 = \frac{qE_0}{m(\underbrace{\omega_0^2-\omega^2}_{\alpha}+i\underbrace{\gamma\omega}_{\beta})}, \quad \text{where} \quad x = x_0 e^{i\omega t}$$

$$= \frac{-eE_0}{m}\frac{\alpha-i\beta}{\alpha^2+\beta^2}$$

$$|x_0| = \frac{eE_0}{m}\frac{1}{\alpha^2+\beta^2}\sqrt{\alpha^2+\beta^2} = \frac{eE_0}{m}\frac{1}{\sqrt{\alpha^2+\beta^2}} =$$

$$= \frac{eE_0}{m}\left[\frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right]^{1/2}$$

At resonance, $\omega_0 = \omega$

$$|x_0| = \frac{eE_0}{m}\frac{1}{\gamma\omega_0}$$

$|x_0| = 10^{-10}\,m$ $\quad \gamma = 2\pi \times 10\,GHz = 2\pi \times 10^{10}\,1/sec$

$m = 9.1\times10^{-31}\,kg$ $\quad \omega_0 = 2\pi \cdot 10^{14}\,1/sec$

$e = 1.6\times10^{-19}\,C$

$$E_0 = \frac{m|x_0|}{e}\gamma\omega_0$$

$$E_0 = \frac{9.1\times10^{-31}\,kg \times 10^{-10}\,m}{1.6\times10^{-19}\,C}\,10^{10}\,Hz \times (2\pi)^2 \times 10^{14}\,1/sec$$

$$\boxed{E_0 = 22.5\ kV/m}$$

(b.)

$$\langle I\rangle = P/A = \frac{10\,W}{\frac{\pi}{4}(10^{-4}\,m)^2} = 1.27\times10^{9}\,W/m^2$$

$$\langle I\rangle = \tfrac{1}{2}\epsilon_0 c E_0^2 \rightarrow E_0 = \sqrt{\frac{2\langle I\rangle}{\epsilon_0 c}}$$

$$= \sqrt{\frac{2\cdot1.27\times10^{9}\,W/m^2}{8.85\times10^{-12}\,F/m \times 3\times10^{8}\,m/s}}$$

$$= 9.79\cdot10^{5}\,V/m$$

$$|x_0| = \frac{eE_0}{m}\frac{1}{\gamma\omega_0}$$

$$= \frac{1.6\times10^{-19}\,C}{9.11\times10^{-31}\,kg} \times \frac{6.93\times10^{5}\,V/m}{10^{10}\,Hz\times(2\pi)^2\times10^{14}\,sec^{-1}}$$

$$= 4.36\times10^{-9}\,m = \boxed{4.4\,nm}$$

(c.) $N = 10^{22}\ \text{atoms}/cm^3$

$$K = \frac{Nq^2}{2\epsilon_0 m}\frac{\gamma\omega}{(\omega_0^2-\omega^2)+\gamma^2\omega^2}$$

$$n' = 1 + \frac{Nq^2}{2\epsilon_0 m}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

When $\omega_0 = \omega$:

$$\boxed{n' = 1}$$

$$K = \frac{10^{22}\ \text{atom}/cm^3\times\left(\frac{10^2 cm}{1m}\right)^3\left(1.6\times10^{-19}\,C\right)^2}{2\times8.85\times10^{-12}\,F/m\times9.1\times10^{-31}\,kg}\ \frac{1}{10^{10}\,Hz\times(2\pi)^2\times10^{14}\,Hz}$$

$$\boxed{K = 4.07\times10^5}$$

If you calculate the absorption coefficient, α:

$$\alpha = \frac{4\pi K}{\lambda} = \frac{4\pi\cdot4.1\times10^5}{3\times10^8\,m/s}\,10^{14}\,Hz = 1.7\times10^{12}\,cm^{-1}$$

This absorption coefficient is extremely high: in 1 nm, the absorption will be $\sim10^5$, far higher than any 'real' material.

2.

(a.) $$e^{-iH't/\hbar} = \cos\left(\frac{\Omega t}{2}\right)\underline{\underline{I}} - i\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)\sin\left(\frac{\Omega t}{2}\right)$$

$$H' = \frac{\hbar\Omega}{2}\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)$$

$$e^{-iH't/\hbar} = \cos(H't/\hbar) - i\sin(H't/\hbar)$$

$$= \cos\left(\frac{\Omega t}{2}\left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]\right) - i\sin\left(\frac{\Omega t}{2}\left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]\right)$$

$$\cos\theta\sigma_z + \sin\theta\sigma_x = \begin{bmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{bmatrix} + \begin{bmatrix}0 & \sin\theta\\ \sin\theta & 0\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

Let's start with the cosine term:

$$\cos\Big(\underbrace{\frac{\Omega t}{2}}_{a}\underbrace{\left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]}_{x}\Big) = \cos(ax) = 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} + \dots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(ax)^{2n}}{(2n)!}$$

$$x^0 = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}^0 = \underline{\underline{I}}$$

$$x^2 = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} = \begin{bmatrix}\cos^2\theta + \sin\theta & 0\\ 0 & \sin^2\theta + \cos^2\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \underline{\underline{I}}$$

$$x^{2n} = (x^2)^n = (\underline{\underline{I}})^n = \underline{\underline{I}}$$

$$\cos(ax) = \sum_{n=0}^{\infty}\frac{(-1)^n(ax)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(2n)!}(x^2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(2n)!}\underline{\underline{I}} = \underline{\underline{I}}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(2n)!}$$

$$= \underline{\underline{I}}\cos(a) = \boxed{\cos\left(\frac{\Omega t}{2}\right)\underline{\underline{I}}}$$

Now let's deal w/ the sine term:

$$\sin\Big(\underbrace{\frac{\Omega t}{2}}_{a}\underbrace{\left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]}_{x}\Big) = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}(ax)^{2n-1}$$

$$x' = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$x^3 = \underbrace{x^2}_{\mathbb{I}} \cdot x = x$$

$$x^{2n-1} = x^{2n-2} \cdot x = (x^2)^{n-1} \cdot x = \mathbb{I}^{n-1} \cdot x = x$$

$$\sin(ax) = \sum_{n-1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}(ax)^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} a^{2n-1} x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} a^{2n-1} \mathbb{I} x$$

$$= x\mathbb{I} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} a^{2n-1} = \left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]\sin(a)$$

$$= \boxed{\left[\cos\theta\sigma_z + \sin\theta\sigma_x\right]\sin\left(\frac{\Omega t}{2}\right)}$$

Putting it together, we have:

$$e^{-iH't/\hbar} = \cos\left(\frac{\Omega t}{2}\right)\mathbb{I} - i\left[\cos\theta\vec{\sigma}_z + \sin\theta\vec{\sigma}_x\right]\sin\left(\frac{\Omega t}{2}\right)$$

QED

(b.)
(i.) $\sigma_z |\beta\rangle = -|\beta\rangle$, where $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1+\lambda) = 0$$
$$\lambda = \pm 1$$

$\underline{\lambda = +1}$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |\alpha\rangle = |\uparrow\rangle$$

$\underline{\lambda = -1}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \equiv -|\beta\rangle = -|\downarrow\rangle$$

As such, when σ_z operates on $|\beta\rangle$, we get $-|\beta\rangle$, the eigenvector associated w/ the -1 eigenvalue.

(ii) $\sigma_x |\beta\rangle = |\alpha\rangle$, where $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 1 \longrightarrow \lambda = \pm 1$$

$\underline{\lambda = +1}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \left.\begin{array}{l} -a+b = a \\ a-b = b \end{array}\right\}$$ summing gives: $0 = a+b$, $-b = a$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \equiv |\alpha\rangle_x = |\uparrow\rangle_x$$

$\lambda = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow$$

$a + b = -a$

$a + b = -b$

$0 = -a + b$ (subtracting)

$a = b$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \equiv |\beta\rangle_x = |\downarrow\rangle_x$$

$$\sigma_x |\beta\rangle_z = \sigma_x\left[\frac{\sqrt{2}}{2}|\alpha\rangle_x + \frac{\sqrt{2}}{2}|\beta\rangle_x\right]$$

$$= \frac{\sqrt{2}}{2}\left\{ |\alpha\rangle_x + |\beta\rangle_x \right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |\alpha\rangle_z$$

3.

(a.) The spontaneous and induced transition probabilities are equal when the radiation field contains (on average) one photon per mode.

$$\bar{n} = \frac{1}{e^{h\nu/k_BT} - 1} = 1$$

$$e^{h\nu/k_BT} = 2$$

$$\frac{h\nu}{k_B \ln 2} = T \longrightarrow \text{Since } c/\nu = \lambda, \text{ we have:}$$

$$T = \frac{hc}{k_B \lambda \ln 2}, \quad \text{where } \lambda = 589\text{ nm}$$

$k_B = 86.17\ \mu eV/K$

$hc = 1240\ \text{nm-eV}$

$$T = \frac{1240\ \text{nm-eV}}{8.617\times10^{-5}\ eV/K\,(589\text{nm})\ln 2}$$

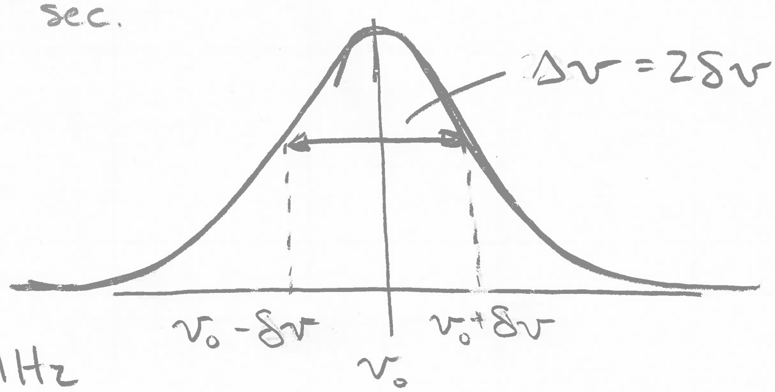
$$\boxed{T = 3.52\times10^{4}\ K}$$

(b.) $\tau = 1.6\times10^{-8}$ sec.

$$\sigma_E \sigma_t \geq \hbar/2$$

$$h\delta\nu \geq \frac{\hbar}{2\sigma_t}$$

$$\delta\nu \geq \frac{1}{4\pi\sigma_t}$$



$$\delta\nu \geq 4.97\text{ MHz}$$

$$\Delta\nu = 9.95\text{ MHz} = 10\text{ MHz}$$

(c.) $n\,d\nu = \frac{8\pi}{c\lambda^2}d\nu$ = # of modes in the cavity w/in the freq. interval, $\Delta\nu$

$$\frac{8\pi\,(10^7\text{ Hz})}{(3\cdot10^8 m/s)(589\cdot10^{-7} m)^2} = 2.4\times10^{12}\ m^{-3}$$

$$= \boxed{2.4\times10^{6}\ cm^{-3}}$$

(d.)

Energy of one photon @ 589 nm:

$$\frac{1240 \text{ nm-eV}}{589 \text{ nm}} \times \frac{1.602\times10^{-19}\text{ J}}{1\text{ eV}} = 3.37\times10^{-19}\text{ J} = 3.37\times10^{-19}\text{ W-s}$$

Since there is only one photon per mode in the cavity ($\bar{n} = 1$), the radiation density is:

$$\rho = \underbrace{\bar{W} n\,d\nu}_{\text{part (c.)}}$$, where $\bar{W}$ = avg. photon energy in a mode

$\bar{W} = h\nu$

and $n\,d\nu$ = # of modes per unit volume in a freq. interval $d\nu$

$$\rho = 3.37\times10^{-19}\text{ W-s}\,(2.4\times10^{6}\text{ cm}^{-3}) = \boxed{8.09\times10^{-13}\ \frac{\text{W-s}}{\text{cm}^3}}$$

ρ is the radiation density in the cavity over the $\Delta\nu$ frequency interval

(e.)

$$I = \rho c$$

$$= \left(8.09\times10^{-13}\ \frac{\text{W-s}}{\text{cm}^3}\right)\left(3\times10^{10}\text{ cm/s}\right)$$

$$= 2.4\times10^{-2}\ \text{W/cm}^2 = \boxed{24\ \text{mW/cm}^2}$$