PHYS 4310, Quiz 5 November 20, 2015

The Fourier transform of $f(\vec{x})$ in three dimensions can be defined as:

$$F\left(\vec{k}\right) = \sqrt{\frac{1}{2\pi}} \int_V f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}} dV, \tag{1}$$

where $\vec{k}$ and $\vec{x}$ are vectors in 3D.

Assuming $f(\vec{x})$ only depends on r, obtain $F\left(\vec{k}\right)$ as an integral in terms of $f(r)$ and r, where r is the radial variable in spherical coordinates.

$$F(\bar{k}) = \sqrt{\frac{1}{2\pi}} \int_0^{2\pi} d\phi \int_0^{\infty}\int_0^{\pi} F(r) e^{-ikr\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \sqrt{2\pi} \int_0^{\infty}\int_0^{\pi} f(r) e^{-ikr\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\left\{ \begin{array}{l} \cos\pi = -1 \\ \cos 0 = +1 \end{array} \right.$$

$$= \sqrt{2\pi} \int_0^{\infty}\int_{-1}^{1} f(r) e^{-ikru} r^2 \, du\, dr$$

$$= \sqrt{2\pi} \int_0^{\infty} f(r) \left[\frac{1}{-ikr} \left(e^{-ikru} \right)\Big|_{-1}^{+1} \right] r^2 dr$$

$$= \sqrt{2\pi} \int_0^{\infty} f(r) \left[\frac{e^{+ikr} - e^{-ikr}}{ikr} \right] r^2 dr$$

$$\frac{e^{+i\theta} - e^{-i\theta}}{2i} = \sin\theta$$

$$= 2\sqrt{2\pi} \int_0^{\infty} f(r) \left(\frac{\sin kr}{kr} \right) r^2 dr$$