4.9

$$V(r) = \begin{cases} -V_0 & \text{if } r \le r_0 \\ 0 & \text{if } r > r_0 \end{cases}$$

Just as in the example, $u(r) = rR(r) = A\sin(kr)$

However, $k \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$, instead of $\frac{\sqrt{2mE}}{\hbar}$

For $r > r_0$, $V=0$ and $E<0$ (bound state)

$$\frac{d^2u}{dr^2} = \left[\frac{\ell(\ell+1)}{r^2} + \alpha^2\right]u \quad , \alpha = \frac{\sqrt{-2mE}}{\hbar}$$

As the problem states, we are only concerned about the $\ell=0$ case

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\cancel{V}^{\to 0} + \frac{\hbar^2}{2m}\cancel{\frac{\ell(\ell+1)}{r^2}}^{\to 0}\right]u = Eu$$

$$\frac{d^2u}{dr^2} = \frac{-2mE}{\hbar^2}u = \alpha^2 u \rightarrow u(r) = Ce^{\alpha r} + De^{-\alpha r}$$

As $r \to \infty$, $Ce^{\alpha r} \to \infty$, so $C=0$

$$u(r) = \begin{cases} A\sin kr & r \le r_0 \\ De^{-\alpha r} & r \ge r_0 \end{cases}$$

Continuity of $u(r)$ @ $r = r_0$

① $A\sin kr_0 = De^{-\alpha r_0}$

Continuity of $\frac{du}{dr}$ @ r_0

② $Ak\cos kr_0 = -D\alpha e^{-\alpha r_0}$

Dividing ① by ②:

$$\frac{1}{k}\tan kr_0 = -\frac{1}{\alpha} \rightarrow -\frac{\alpha}{k} = \cot kr_0$$

Same as before: $kr_0 \equiv z \rightarrow \frac{\alpha}{k} = \frac{r_0\sqrt{-2mE/\hbar^2}}{ka}$

$$\frac{\alpha}{k} = \frac{r_0}{k r_0}\sqrt{-\frac{2mE}{\hbar^2}} = \frac{r_0}{\underbrace{k r_0}_{z}}\sqrt{\frac{2mV_0}{\hbar^2} - \underbrace{\frac{2m(E+V_0)}{\hbar^2}}_{k^2}}$$

$$= \frac{1}{z}\sqrt{\boxed{\frac{2mV_0}{\hbar^2}r_0^2} - z^2}$$

let's define $z_0 \equiv \frac{\sqrt{2mV_0}}{\hbar}r_0$, so that $z_0^2 = \frac{2mV_0}{\hbar^2}r_0^2$

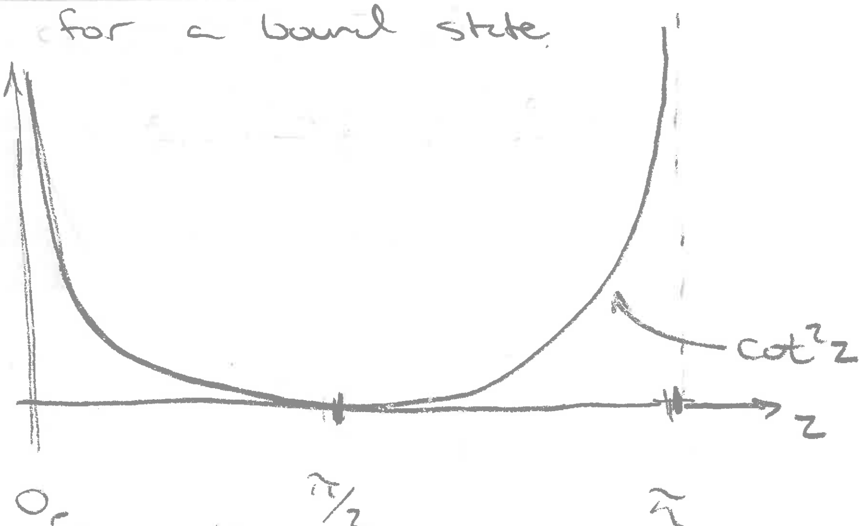
$$\frac{\alpha}{k} = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

$$\therefore \quad -\cot z = \frac{\alpha}{k} = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

Let's find the condition for a bound state

$$-\cot z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

$$\cot^2 z + 1 = \left(\frac{z_0}{z}\right)^2$$



Lowest point is @ $\pi/2$ $\left[\left(\frac{\cos(\pi/2)}{\sin(\pi/2)}\right)^2 + 1 = 0 + 1 = 1\right]$ so:

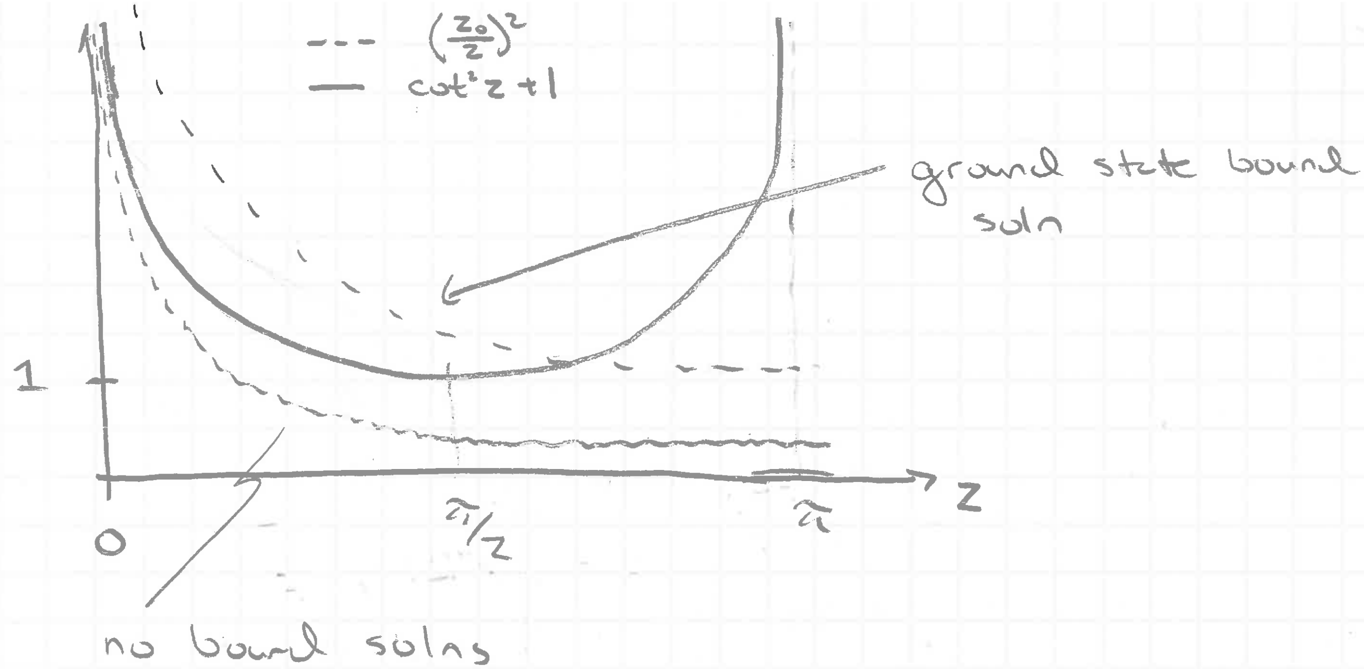
the point below which we no longer obtain bound solns is $z = \pi/2$

$$\pm 1 = \frac{z_0}{z} \quad \rightarrow \quad \frac{\pi}{2} > z_0$$ (if less than this, no bound solns)

$$\frac{\pi}{2} > \frac{\sqrt{2mV_0}}{\hbar}r_0$$

$$\left(\frac{\hbar\pi}{2r_0}\right)^2 \frac{1}{2m} > V_0$$

$$\frac{\pi^2\hbar^2}{8m} > V_0 r_0^2$$



The ground state bound soln occurs between $z = \pi/2$ and π.

$$z^2 = k^2 r_0^2 = \frac{2m(E+V_0)}{\hbar^2} r_0^2$$

$$\left(\frac{\pi}{2}\right)^2 < \frac{2m(E+V_0)}{\hbar^2} r_0^2 < \pi^2$$

$$\frac{\pi^2\hbar^2}{8mr_0^2} < E + V_0 < \frac{\pi^2\hbar^2}{2mr_0^2}$$

4.11

(a.)

$$R_{20}(r) = \frac{c_0}{2a_0}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}$$

$$1 = \int_0^\infty |R_{20}(r)|^2 r^2\, dr = \frac{|c_0|^2}{4a_0^2} \int_0^\infty e^{-\frac{2r}{2a_0}} \left(1 - \frac{2r}{2a_0} + \frac{r^2}{4a_0^2}\right) r^2$$

$$= \frac{|c_0|^2}{4a_0^2}\left[\underbrace{\int_0^\infty r^2 e^{-\frac{r}{a_0}}\, dr}_{①} - \frac{1}{a_0}\underbrace{\int_0^\infty r^3 e^{-\frac{r}{a_0}}\, dr}_{②} + \frac{1}{4a_0^2}\underbrace{\int_0^\infty r^4 e^{-\frac{r}{a_0}}\, dr}_{③}\right]$$

① IBP

r^2	$\oplus$	e^{-r/a_0}
$2r$	$\ominus$	$-a_0 e^{-r/a_0}$
2	$\oplus$	$a_0^2 e^{-r/a_0}$
0		$-a_0^3 e^{-r/a_0}$

$$\int_0^\infty r^2 e^{-r/a_0}\, dr = -a_0 e^{-r/a_0}\left[r^2 + 2ra_0 + 2a_0^2\right]\Big|_0^\infty$$

$$= 2a_0^3$$ (all terms @ $r=\infty$ evaluate to 0 and r^2+2ra_0 are 0 @ $r=0$)

② IBP

r^3	$\oplus$	e^{-r/a_0}
$3r^2$	$\ominus$	$-a_0 e^{-r/a_0}$
$6r$	$\oplus$	$a_0^2 e^{-r/a_0}$
6	$\ominus$	$-a_0^3 e^{-r/a_0}$
0		$a_0^4 e^{-r/a_0}$

$$\int_0^\infty r^3 e^{-r/a_0}\, dr = \left[r^3 + 3r^2a_0 + 6ra_0^2 + 6a_0^3\right]\left(-a_0 e^{-r/a_0}\right)\Big|_0^\infty$$

$$= 6a_0^4$$ (again, no non-zero terms @ $r=\infty$ and only one @ $r=0$)

③ IBP

r^4	$\oplus$	e^{-r/a_0}
$4r^3$	$\ominus$	$-a_0 e^{-r/a_0}$
$12r^2$	$\oplus$	$a_0^2 e^{-r/a_0}$
$24r$	$\ominus$	$-a_0^3 e^{-r/a_0}$
24	$\oplus$	$a_0^4 e^{-r/a_0}$
0		$-a_0^5 e^{-r/a_0}$

$$\int_0^\infty r^4 e^{-r/a_0}\, dr = -a_0 e^{-r/a_0}\left[r^4 + 4r^3a_0 + 12r^2a_0^2 + 24ra_0^3 + 24a_0^4\right]\Big|_0^\infty$$

$$= 24a_0^5$$

$$1 = \frac{|c_0|^2}{4a_0^2}\left[2a_0^3 - \frac{6\cancel{a_0^4}^{\,3}}{\cancel{a_0}} + \frac{1}{\cancel{4a_0^2}}\,\overset{6}{\cancel{24}}\,\cancel{a_0^5}^{\,3}\right]$$

$$= \frac{|c_0|^2}{4}\left[2a_0 - \cancel{6a_0} + \cancel{6a_0}\right] = \frac{|c_0|^2}{2}a_0$$

$$\boxed{\sqrt{\frac{2}{a_0}} = c_0}$$

$$R_{20} = \sqrt{\frac{2}{a_0}}\,\frac{1}{2a_0}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0} = \boxed{\sqrt{\frac{1}{2a_0^3}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}}$$

$$\psi_{200} = R_{20}\underbrace{\boxed{Y_0^0}}_{\sqrt{\frac{1}{4\pi}}} = \boxed{\sqrt{\frac{1}{(2a_0)^3\pi}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}}$$

(b.) $$R_{21}(r) = \frac{c_0}{4a_0^2}\,r e^{-r/2a_0} \Rightarrow 1 = \int_0^\infty r^2 R_{21}^2(r)\,dr$$

$$1 = \frac{|c_0|^2}{16a_0^4}\underbrace{\int_0^\infty r^4 e^{-r/a_0}\,dr}_{\text{see result from (a.)}} = \frac{|c_0|^2}{16a_0^4}\left[24a_0^5\right] = \frac{3|c_0|^2}{2}a_0$$

$$\boxed{\sqrt{\frac{2}{3a_0}} = c_0}$$

$$R_{21}(r) = \sqrt{\frac{2}{3a_0}}\,\frac{1}{4a_0^2}\,r e^{-r/2a_0} = \boxed{\sqrt{\frac{1}{6a_0}}\,\frac{1}{2a_0^2}\,r e^{-r/2a_0}}$$

$$Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

$$\psi_{211} = R_{21}(r)\boxed{Y_1^1(\theta,\phi)} = \sqrt{\frac{2}{3a_0}}\,\frac{1}{4a_0^2}\,r e^{-r/2a_0}\left[-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right]$$

$$\psi_{210} = R_{21}(r)\underbrace{Y_1^0(\theta,\phi)}_{\sqrt{\frac{3}{4\pi}}\cos\theta} = \sqrt{\frac{2}{3a_0}}\,\frac{1}{4a_0^2}\,r e^{-r/2a_0}\left[\sqrt{\frac{3}{4\pi}}\cos\theta\right]$$

$$\psi_{21\bar{1}} = R_{21}(r)\boxed{Y_1^{-1}(\theta,\phi)} = \sqrt{\frac{2}{3a_0}}\,\frac{1}{4a_0^2}\,r e^{-r/2a_0}\left[\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}\right]$$

$$\psi_{211}(r,\theta,\phi) = \frac{-1}{8a_0^2}\sqrt{\frac{1}{\pi a_0}}\, r e^{-r/2a_0} \sin\theta\, e^{i\phi}$$

$$\psi_{210}(r,\theta,\phi) = \frac{1}{4a_0^2}\sqrt{\frac{1}{2\pi a_0}}\, r e^{-r/2a_0} \cos\theta$$

$$\psi_{21\bar{1}}(r,\theta,\phi) = \frac{1}{8a_0^2}\sqrt{\frac{1}{\pi a_0}}\, r e^{-r/2a_0} \sin\theta\, e^{-i\phi}$$

4.12

(a.) $L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q (e^{-x} x^q)$

$q=0$ $\quad L_0(x) = e^x \left(\frac{d}{dx}\right)^0 (e^{-x} \cancel{x^0}) = e^x e^{-x} = 1$

$q=1$ $\quad L_1(x) = e^x \left(\frac{d}{dx}\right)(e^{-x} x) = e^x\left[-e^{-x}x + e^{-x}\right] = 1-x$

$q=2$ $\quad L_2(x) = e^x \left(\frac{d^2}{dx^2}\right) e^{-x} x^2 = e^x\left[\frac{d}{dx}\left(-e^{-x}x^2 + 2xe^{-x}\right)\right]$

$$= e^x\left[e^{-x}x^2 - 2xe^{-x} + 2e^{-x} - 2xe^{-x}\right]$$

$$= x^2 - 2x + 2 - 2x = \boxed{x^2 - 4x + 2}$$

$q=3$ $\quad L_3(x) = e^x \frac{d^3}{dx^3}\left[e^{-x}x^3\right] = e^x\left[\frac{d^2}{dx^2}\left(-e^{-x}x^3 + 3x^2e^{-x}\right)\right]$

$$= e^x\left[\frac{d}{dx}\left(e^{-x}x^3 - 3x^2e^{-x} + 6xe^{-x} - 3x^2e^{-x}\right)\right]$$

$$= e^x\left[-e^{-x}x^3 + 3x^2e^{-x} - 6xe^{-x} + 3x^2e^{-x} + 6e^{-x} - 6xe^{-x} - 6xe^{-x} + 3x^2e^{-x}\right]$$

$$= -x^3 + 3x^2 - 6x + 3x^2 + 6 - 6x - 6x + 3x^2$$

$$= \boxed{-x^3 + 9x^2 - 18x + 6}$$

(b.) $n=5$, $l=2$

$$v(\rho) = L^{2l+1}_{n-l-1}(2\rho) \rightarrow L^{5}_{2}\left(\frac{2r}{5a_0}\right) \quad \text{where } \rho = \frac{r}{na_0}$$

$$L^{p}_{q-p}(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_q(x) \rightarrow L^{5}_{2}\left(\frac{2r}{5a_0}=x\right) = (-1)^5 \frac{d^5}{dx^5} L_7(x)$$

$$L_q(x) \equiv e^{x}\left(\frac{d}{dx}\right)^q (e^{-x}x^q)$$

$$L_7(x) = e^{x}\left[\frac{d^7}{dx^7}(e^{-x}x^7)\right] = e^{x}\left[\frac{d^6}{dx^6}\left(-e^{-x}x^7 + 7x^6e^{-x}\right)\right]$$

$$= e^{x}\left[\frac{d^5}{dx^5}\left(e^{-x}x^7 - 7x^6e^{-x} + 42x^5e^{-x} - 7x^6e^{-x}\right)\right]$$

$$= e^{x}\left[\frac{d^4}{dx^4}\underbrace{\left(-e^{-x}x^7 + 7e^{-x}x^6 - 42x^5e^{-x} + 7x^6e^{-x} + 210x^4e^{-x} - 42x^5e^{-x} - 42x^5e^{-x} + 7x^6e^{-x}\right)}_{-e^{-x}x^7 + 21e^{-x}x^6 - 126e^{-x}x^5 + 210e^{-x}x^4}\right]$$

$$= e^{x}\left[\frac{d^3}{dx^3}\left(e^{-x}x^7 - 7x^6e^{-x} - 21e^{-x}x^6 + \underbrace{126e^{-x}x^5 + 126e^{-x}x^5}_{252e^{-x}x^5} - 630e^{-x}x^4 - 210e^{-x}x^4 + 840e^{-x}x^3\right)\right]$$

$$= e^{x}\left[\frac{d^2}{dx^2}\left(-e^{-x}x^7 + 7x^6e^{-x} + 7x^6e^{-x} - 42x^5e^{-x} + 21e^{-x}x^6 - 126e^{-x}x^5 - 252e^{-x}x^5 + 1260e^{-x}x^4 + 630e^{-x}x^4 - 2520e^{-x}x^3 + 210e^{-x}x^4 - 840e^{-x}x^3 - 840e^{-x}x^3 + 2520e^{-x}x^2\right]\right.$$

$$= e^{x}\left[\frac{d^2}{dx^2}\left(-e^{-x}x^7 + 35x^6e^{-x} - 420x^5e^{-x} + 2100x^4e^{-x} - 4200x^3e^{-x} + 2520e^{-x}x^2\right)\right]$$

$$= e^{x}\left[\frac{d}{dx}\left(e^{-x}x^7 - 7x^6e^{-x} + 210x^5e^{-x} - 35x^6e^{-x} - 2100x^4e^{-x} + 420x^5e^{-x} + 8400x^3e^{-x} - 2100x^4e^{-x} - 12600x^2e^{-x} + 4200x^3e^{-x} + 5040xe^{-x} - 2520x^2e^{-x}\right)\right]$$

$$= e^{x}\Big[-e^{-x}x^7 + 7x^6e^{-x} + 7x^6e^{-x} - 42x^5e^{-x} - 210x^5e^{-x} + 1050x^4e^{-x} + 35x^6e^{-x} -$$
$$210x^5e^{-x} + 2100x^4e^{-x} - 8400x^3e^{-x} - 420x^5e^{-x} + 2100x^4e^{-x} - 8400x^3e^{-x} +$$
$$25200x^2e^{-x} + 2100x^4e^{-x} - 8400x^3e^{-x} + 12600x^2e^{-x} - 25200xe^{-x} - 4200x^3e^{-x}$$
$$+ 12600x^2e^{-x} - 5040xe^{-x} + 5040e^{-x} + 2520x^2e^{-x} - 5040xe^{-x}\Big]$$

$$L_7(x) = -x^7 + 49x^6 - 882x^5 + 7350x^4 - 29400x^3 + 52920x^2 - 35280x + 5040$$

$L_2^5(x) = (-1)^5 \frac{d^5}{dx^5} L_7(x)$ ⟹ b/c we are taking the fifth derivative wrt x, any power below x^5 will be zero.

$$(-1)\left[-7\cdot6\cdot5\cdot4\cdot3x^2 + 49(6\cdot5\cdot4\cdot3\cdot2)x - 882(5\cdot4\cdot3\cdot2)\right]$$

$$L_2^5(x) = -\underbrace{(5\cdot4\cdot3)}_{60}\left[-42x^2 + \underset{14}{588x} - 1764\right] = 2520(x^2 - 14x + 42)$$

$$v(\rho) = L_2^5\left(\tfrac{2r}{5a_0} = 2\rho\right) = 2520(4\rho^2 - 28\rho + 42) = 2520\left(\frac{4r^2}{25a_0^2} - \frac{28r}{5a_0} + 42\right)$$
$$= 5040(2\rho^2 - 14\rho + 21).$$

(c.)

$$c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j \qquad \left.\begin{array}{l} j_{max} = n-l-1 \\ n=5,\ l=2 \end{array}\right] \Rightarrow j_{max} = 2$$

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

$j=0$

$$c_1 = \frac{2(0+2+1-5)}{(1)(0+4+2)} = \frac{-4}{6}c_0 = -\frac{2}{3}c_0$$

$j=1$ $$C_2 = \frac{2(1+2+1-5)}{2(1+4+2)} C_1 = \frac{-2}{2(7)} C_1 = \frac{C_1}{7}$$

$$C_2 = \frac{-C_1}{7} = \frac{2}{21} C_0$$

$j=2$ $$C_3 = \frac{2(2+2+1-5)}{3(2+4+2)} C_2 = 0$$

$$v(\rho) = \sum_{j=0}^{\infty} C_j \rho^j = C_0 + C_1\rho + C_2\rho^2 = C_0\left(1 - \frac{2}{3}\rho + \frac{2}{21}\rho^2\right)$$

$$\boxed{v(\rho) = \frac{C_0}{21}\left(21 - 14\rho + 2\rho^2\right)} \longrightarrow \text{same scaling as (b.)}$$

4.13 $$\psi_{100}(r,\theta,\phi) = \sqrt{\frac{1}{\pi a_0^3}}\, e^{-r/a_0}$$

$$\langle r \rangle = \iiint r^2 \sin\theta \, dr\, d\theta\, d\phi \; r|\psi_{100}|^2 = \frac{4\pi}{\pi a_0^3}\int_0^\infty r^3 e^{-2r/a_0}\, dr$$

from angular integrations

$$= \frac{4}{a_0^3}\int_0^\infty r^3 e^{-2r/a_0}\, dr \Rightarrow \int_0^\infty r^3 e^{-2r/a_0}\, dr$$

r^3	$\oplus$	e^{-2r/a_0}
$3r^2$	$\ominus$	$-\frac{a_0}{2}e^{-2r/a_0}$
$6r$	$\oplus$	$\frac{a_0^2}{4}e^{-2r/a_0}$
6	$\ominus$	$-\frac{a_0^3}{8}e^{-2r/a_0}$
0		$\frac{a_0^4}{16}e^{-2r/a_0}$

$$= \frac{4}{a_0^3}\left[\frac{-a_0}{2}e^{-2r/a_0}\left(r^3 + \frac{3}{2}r^2 a_0 + \frac{3}{2}r a_0^2 + \frac{3}{4}a_0^3\right)\right]_0^\infty$$

$$= \frac{4\cdot 3}{\cancel{a_0^3}\,\cancel{4}}\cancel{a_0^3}\left(\frac{a_0}{2}\right) = \boxed{\frac{3a_0}{2}}$$

$$\langle r^2 \rangle = \frac{4\pi}{\pi a_0^3}\int_0^\infty r^4 e^{-2r/a_0}\, dr = \frac{4}{a_0^3}\int_0^\infty r^4 e^{-2r/a_0}\, dr$$

$$\int_0^\infty r^4 e^{-2r/a_0}\, dr$$

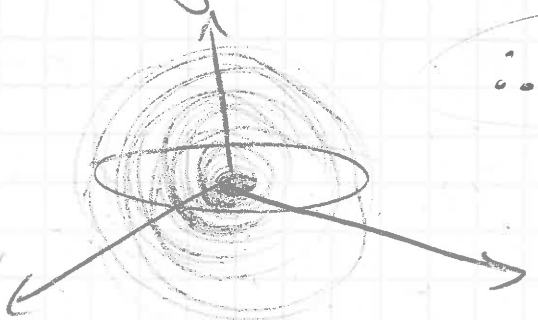
	sign	
r^4	$+$	e^{-2r/a_0}
$4r^3$	$-$	$-\frac{a_0}{2} e^{-2r/a_0}$
$12r^2$	$+$	$\frac{a_0^2}{4} e^{-2r/a_0}$
$24r$	$-$	$-\frac{a_0^3}{8} e^{-2r/a_0}$
24	$+$	$\frac{a_0^4}{16} e^{-2r/a_0}$
0		$-\frac{a_0^5}{32} e^{-2r/a_0}$

Taking into account only the last term $\left(-\frac{24}{32} a_0^5 e^{-2r/a_0}\right)$ evaluated @ $r=0$ and ∞ gives: $\frac{3}{4} a_0^5$

$$\langle r^2 \rangle = \frac{4}{a_0^3} \frac{3}{4} a_0^5 = 3a_0^2$$

(b.) The ground state for ^1H is centered on $(x,y,z) = (0,0,0)$

$\therefore \langle x \rangle = \langle y \rangle = \langle z \rangle = 0$



$\langle r^2 \rangle = 3\langle x^2 \rangle = 3\langle y^2 \rangle = 3\langle z^2 \rangle$ so $\langle x^2 \rangle = \frac{\langle r^2 \rangle}{3} = a_0^2$

(c.) First, let's calculate $\psi_{211}(r,\theta,\phi) = R_{21}(r) Y_1^1(\theta,\phi)$

$$\psi_{211}(r,\theta,\phi) = \frac{-1}{8a_0^2}\sqrt{\frac{1}{\pi a_0}}\, r e^{-r/2a_0} \sin\theta\, e^{i\phi}$$ [problem 4.11 (b.)]

$x = r\sin\theta\cos\phi$

$$\langle x^2 \rangle = \int_V \psi_{211}^* x^2 \psi_{211}\, dV = \frac{1}{64a_0^4}\frac{1}{\pi a_0} \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 e^{-\frac{2r}{2a_0}} \sin^2\theta\, r^2 \sin^2\theta \cos^2\phi\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\langle x^2\rangle = \frac{1}{64\pi a_0^5}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} r^6 e^{-\frac{r}{a_0}} \sin^5\theta \cos^2\phi \, dr\, d\theta\, d\phi$$

ϕ:

$$\int_0^{2\pi} \cos^2\phi \, d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2} d\phi = \frac{1}{2}\left(2\pi + \cancel{\frac{\sin 2\phi}{4}\Big|_0^{2\pi}}^{\,0}\right) = \pi$$

θ:

$$\int_0^{\pi} \sin^5\theta \, d\theta = \int_0^{\pi} (\underbrace{1-\cos^2\theta}_{\sin^2\theta})^2 \sin\theta\, d\theta = \int_0^{\pi} (1 - 2\cos^2\theta + \cos^4\theta)\sin\theta\, d\theta$$

$$= \int_0^{\pi} \sin\theta\, d\theta - 2\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta + \int_0^{\pi} \cos^4\theta \sin\theta\, d\theta.$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$$= -\cos\theta\Big|_0^{\pi} + 2\int_1^0 x^2\, dx - \int_1^0 x^4\, dx = -(-1-1) + \frac{2}{3}\cos^3\theta\Big|_0^{\pi} - \frac{1}{5}\cos^5\theta\Big|_0^{\pi}$$

$$= 2 + \frac{2}{3}(-1-1) - \frac{1}{5}(-1-1)$$

$$= 2 - \frac{4}{3} + \frac{2}{5} = \frac{30-20+6}{15} = \frac{16}{15}$$

r:

$$\int_0^{\infty} r^6 e^{-r/a_0}\, dr = 6!\, a_0^7 \quad \text{(from formula in back of book)}$$

$$\langle x^2\rangle = \frac{1}{\cancel{4}\cancel{64}\cancel{\pi}\cancel{a_0^5}} \underset{r}{6!\, a_0^{\cancel{7}\,2}} \underset{\theta}{\frac{\cancel{16}}{5\cdot 3}} \underset{\phi}{\cancel{\pi}} = \frac{6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}} a_0^2 = 12 a_0^2$$